Vector Models and Vector Spaces

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Fixed Vector Model ————————————————————————————

A **fixed vector** $\vec{X}$ is a one-dimensional array called a **column vector** or a **row vector**, denoted correspondingly by

$$(1) \qquad \vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{or} \quad \vec{X} = (x_1, x_2, \ldots, x_n).$$

The **entries** or **components** $x_1, \ldots, x_n$ are numbers and n is correspondingly called the **column dimension** or the **row dimension** of the vector in (1). The set of all n-vectors (1) is denoted R^n.

Vector addition and **vector scalar multiplication** are defined by componentwise operations as follows.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad k \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{pmatrix}.$$

The Mailbox Analogy

Fixed vectors can be visualized as in Table 1, in which the fixed vector entries $x_1, \ldots, x_n$ appear as contents of mailboxes with names $1, 2, \ldots, n$.

Table 1. The mailbox analogy: box i has contents x_i.

x_1	mailbox 1
x_2	mailbox 2
$\vdots$	$\vdots$
x_n	mailbox n

Free Vector Model ⸻

In the model, **rigid motions** from geometry are applied to directed line segments. A line segment $\overline{PQ}$ is represented as an **arrow** with head at Q and tail at P. Two such arrows are considered **equivalent** if they can be **rigidly translated** to the same arrow whose tail is at the origin. The arrows are called **free vectors**. They are denoted by the symbol $\overrightarrow{PQ}$ or sometimes $\vec{A}$, which labels the arrow with tail at P and head at Q.

Vector Addition ⸻

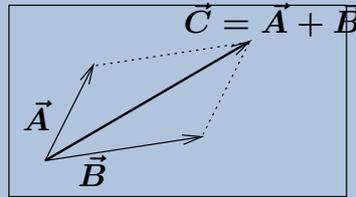The parallelogram rule defines **free vector addition**, as in Figure 1.

Figure 1. Free vector addition. The diagonal of the parallelogram formed by free vectors $\vec{A}$, $\vec{B}$ is the sum vector $\vec{C} = \vec{A} + \vec{B}$.

Scalar Multiplication

To define **free vector scalar multiplication** $k\vec{A}$, we change the location of the head of vector $\vec{A}$; see Figure 2. If $0 < k < 1$, then the head shrinks to a location along the segment between the head and tail. If $k > 1$, then the head moves in the direction of the arrowhead. If $k < 0$, then the head is reflected along the line and then moved.
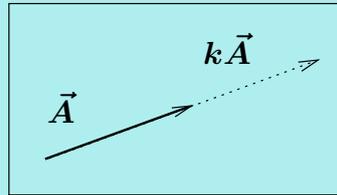

Figure 2. Free vector scalar multiplication. To form $k\vec{A}$, the head of free vector $\vec{A}$ is moved to a new location along the line formed by the head and tail.

Physics Vector Model ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

This model is also called the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ **vector model** and the **orthogonal triad model**. The model arises from the free vector model by inventing symbols $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ for a mutually orthogonal triad of free vectors. Usually, these three vectors represent free vectors of unit length along the coordinate axes, although use in the literature is not restricted to this specialized setting; see Figure 3.
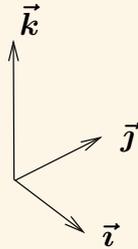
$$\vec{k}$$
$$\vec{\jmath}$$
$$\vec{\imath}$$

Figure 3. Fundamental triad. The free vectors $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ are $90°$ apart and of unit length.

Advantages of the Physics Vector Model ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

The advantage of the model is that any free vector can be represented as $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ for some constants a, b, c, which gives an immediate connection to the free vector with head at (a, b, c) and tail at $(0, 0, 0)$, as well as to the fixed vector whose components are a, b, c.

Vector addition and scalar multiplication are defined componentwise: if $\vec{A} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$, $\vec{B} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ and c is a constant, then

$$\vec{A} + \vec{B} = (a_1 + b_1)\vec{\imath} + (a_2 + b_2)\vec{\jmath} + (a_3 + b_3)\vec{k},$$
$$c\vec{A} = (ca_1)\vec{\imath} + (ca_2)\vec{\jmath} + (ca_3)\vec{k}.$$

Fixed Vectors and Physics Vectors

Formally, computations involving the **physics model** amount to fixed vector computations and the so-called *equalities* between free vectors and fixed vectors:

$$\vec{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Gibbs Vector Model. The model assigns physical properties to vectors, thus avoiding the pitfalls of free vectors and fixed vectors. Gibbs defines a vector as a **linear motion** that takes a point A into a point B. Visualize this idea as a workman who carries material from A to B: the material is loaded at A, transported along a straight line to B, and then deposited at B. Arrow diagrams arise from this idea by representing a motion from A to B as an arrow with tail at A and head at B.

Vector addition is defined as composition of motions: material is loaded at A and transported to B, then loaded at B and transported to C. Gibbs' idea in the plane is the parallelogram law; see Figure 4.
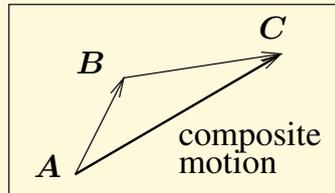

Figure 4. Planar composition of motions. The motion A to C is the composition of two motions or the *sum* of vectors AB and BC.

Gibbs' definitions ⸻

Vector scalar multiplication is defined so that 1 times a motion is itself, 0 times a motion is no motion and -1 times a motion loads at B and transports to A (the reverse motion). If $k > 0$, then k times a motion from A to B causes the load to be deposited at C instead of B, where k is the ratio of the lengths of segments $\overline{AC}$ and $\overline{AB}$. If $k < 0$, then the definition is applied to the reverse motion from B to A using instead of k the constant $|k|$. Briefly, the load to be deposited along the direction to B is dropped earlier if $0 < |k| < 1$ and later if $|k| > 1$.

Vector Spaces and the Toolkit

Consider any vector model: fixed, free, physics or Gibbs. Let V denote the **data set** of one of these models. The data set consists of packages of data items, called **vectors**. Assume a particular dimension, n for fixed, 2 or 3 for the others. Let k, k_1, k_2 be constants. Let $\vec{X}$, $\vec{Y}$, $\vec{Z}$ represent three vectors in V. The following **toolkit** of eight (8) vector properties can be verified from the definitions.

Closure The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new data item package [a vector] which is also in V.

Addition $\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$ commutative
$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$ associative
Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$ zero
Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$ negative

Scalar $k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$ distributive I
multiply $(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$ distributive II
$k_1(k_2\vec{X}) = (k_1 k_2)\vec{X}$ distributive III
$1\vec{X} = \vec{X}$ identity

If you think vectors are arrows, then re-tool your thoughts. Think of vectors as **data item packages**. A technical word, **vector** can also mean a graph, a matrix for a digital photo, a sequence, a signal, an impulse, or a differential equation solution .

Definition 1 (Vector Space)

A data set V equipped with ⊞ and ⊡ operations satisfying the closure law and the eight toolkit properties is called an **abstract vector space**.

The data in V consists of *packages* of data items called **vectors**. The operations ⊞ and ⊡ apply to vectors, that is, to whole packages of data.

The toolkit provides 8 rules to manipulate the packages, without dealing with the inner workings of data items that might be bundled away inside the package. It is usually counterproductive to burst vectors into smaller component pieces. The toolkit hides details and thus encourages data manipulation from its 8 rules.

What's a *Space*?

There is no intended geometrical implication in this term. The usage of **space** originates from phrases like **parking space** and **storage space**.

An abstract vector space is a data set for an application, organized as packages of data items called **vectors**, together with ⊞ and ⊡ operations, which satisfy the toolkit of eight manipulation rules.

The packaging of individual data items is structured, or organized by some scheme, which amounts to a *storage space*, hence the term *space*.

What does *abstract* mean?

The technical details of the packaging and the organization of the data set are invisible to the toolkit rules. The toolkit acts on the formal data packages, which are called **vectors**. Briefly, the toolkit is used **abstractly**, devoid of any details of the storage scheme or internal structure of the data set.