Week 4 Examples

Example 1: Solve the system or explain why it cannot be solved.

$$\begin{cases} x+2y &= 1 \\ x-y &= -2 \end{cases} \qquad \begin{cases} x+2y &= 1 \\ x+2y &= 2 \end{cases} \qquad \begin{cases} x+2y &= 1 \\ 0 &= 0 \end{cases}$$

Answers. Unique solution $x=-1, y=1$; No solution, parallel lines; Infinitely many solutions, parametric equation of the line: $x = 1-2t_1, y = t_1$.

Example 2: Consider $y'' - 121y = 0$, $y(0) = 44$, $y'(0) = 22$. The supplied general solution is $y = Ae^{11x} + Be^{-11x}$. Substitute y into relations $y(0) = 44, y'(0) = 22$ to obtain a 2×2 system for unknowns A, B. Solve it for the unique solution $A = 23, B = 21$.

Example 3: Determine the lead and free variables. Solve the echelon system.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10, \\ \qquad\qquad x_3 + 2x_5 = -3, \\ \qquad\qquad\qquad x_4 - 4x_5 = 7 \end{cases}$$

Answer: Lead $= x_1, x_3, x_4$; Free $= x_2, x_5$. Solution $x_1 = 5+2t_1-3t_2, x_2 = t_1, x_3 = -3-2t_2, x_4 = 7+4t_2, x_5 = t_2$. Method: Assign invented symbols t_1, t_2 to the free variables. Then back-substitute. Best method: Find the reduced echelon system (the *Last Frame*) by Toolkit `combo` applied to the variable list in reverse order. Then apply the *Last Frame Algorithm*: Assign invented symbols to the free variables, isolate the lead variables, then back-substitute. Report for each variable an equation in terms of the invented symbols (the scalar general solution).

Example 4: The unique solution case.

$$\begin{cases} x + 2y + z = 1 \\ x + 3y + z = 2 \\ x + y + 2z = 3 \end{cases}$$

(1) Find the scalar form of the unique solution. (2) Why are there only lead variables in the last frame? (3) Why are there no free variables?

Example 5: The no solution case.

$$\begin{cases} x + 2y + z = 1 \\ x + 3y + z = 2 \\ 2x + 5y + 2z = 4 \end{cases}$$

Use the Toolkit `combo, swap, mult` to find a frame with a signal equation.

Example 6: The infinitely many solution case.

$$\textbf{(6a)} \begin{cases} x + 2y + z = 1 \\ x + 3y + z = 2 \\ 2x + 5y + 2z = 3 \end{cases} \qquad \textbf{(6b)} \begin{cases} x + 2y + z = 1 \\ 3x + 6y + 3z = 3 \\ 0 = 0 \end{cases} \qquad \textbf{(6c)} \begin{cases} 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

(1) Find the last frame or RREF. (2) How many free variables? (3) Apply the last frame algorithm to find the solution.

Example 7: Determine for each value of symbol k the possibility: (1) Unique solution, (2) No solution, (3) Infinitely many solutions.

$$\textbf{(7a)} \begin{cases} x + ky = 2, \\ (2-k)x + y = 3. \end{cases} \qquad \textbf{(7b)} \begin{cases} 3x + 2y = 0, \\ 6x + ky = 0. \end{cases}$$

Example 8: Verify the conversions from scalar equations to augmented matrix, or conversely.

$$\textbf{(8a)} \begin{cases} x + 3y = 2 \\ -x + y = 3 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 1 & 3 \end{array}\right) \qquad \textbf{(8b)} \left(\begin{array}{cc|c} 7 & 5 & 1 \\ -2 & 8 & -1 \end{array}\right) \longrightarrow \begin{cases} 7x + 5y = 1 \\ -2x + 8y = -1 \end{cases}$$

Example 9: Classify as: diagonal, scalar, upper triangular, lower triangular, identity, zero, none of these. More than one classification is possible.

(**9a**) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (**9b**) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (**9c**) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (**9d**) $\begin{pmatrix} \pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e & 1 \end{pmatrix}$

Example 10: Define $\vec{a} = 2\vec{\imath} + 3\vec{\jmath}$, $\vec{b} = -3\vec{\imath} + 2\vec{\jmath}$. Compute the dot product $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$, using the definitions $\vec{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Because of identity $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$, the answer $\vec{a} \cdot \vec{b} = 0$ means the vectors are orthogonal ($\theta = 90$ degrees).

Example 11: Multiply.

(**11a**) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ (**11b**) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & -1 \end{pmatrix}$ (**11c**) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 2 & -1 \end{pmatrix}$

Example 12: Let $A = \begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Compute $C = 2AB + 3B - 5A + 4I$.

Example 13: Write in matrix multiply form $A\vec{x} = \vec{b}$.

$\begin{cases} 7x + 5y = 1 \\ -2x + 8y = -1 \end{cases}$ **Answer**: $\begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Test the answer with **equality of matrices**: two matrices are equal if and only if they have matching entries.

Example 14: An $m \times n$ matrix A is always the coefficient matrix of a homogeneous system $A\vec{x} = \vec{0}$, which represents a scalar system of m equations in n unknowns (the components of $\vec{x}$). The **rank** of A is the number of leading variables. The **nullity** of A is the number of free variables. Always, **rank** + **nullity** = number of variables. Find the rank and nullity.

$\begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix}$, $\begin{pmatrix} 7 & 5 \\ 14 & 10 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 7 & 5 \\ 0 & -2 & 8 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$