Chapter 2

First Order Differential Equations

Contents

The subject of the chapter is the first order differential equation

$$y' = f(x, y).$$

The study includes closed-form solution formulas for special equations, numerical solutions and some applications to science and engineering.

2.1 Quadrature Method

The **method of quadrature** refers to the technique of integrating both sides of an equation, hoping thereby to extract a solution formula.

The term **quadrature** originates in ancient geometry, where it means *finding area* of a plane figure, by constructing a square of equal area.[1] Numerical quadrature computes areas enclosed by plane curves from approximating rectangles, by algorithms such as the rectangular rule

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[1]See Wikipedia, `http://en.wikipedia.org/wiki/Quadrature`.

and Simpson's rule. For symbolic problems, the task is overtaken by Newton's integral calculus. The naming convention follows computer algebra system `maple`.

Fundamental Theorem of Calculus

The foundation of the study of differential equations rests with Isaac Newton's discovery on instantaneous velocities. Details of the calculus background required appears in Appendix A.1, page 839.

Theorem 1 (Fundamental Theorem of Calculus: Definite Integral)
Let G be continuous and let F be continuously differentiable on $[a, b]$. Then

(a) $F(b) - F(a) = \int_a^b \dfrac{dF}{dx}(x)dx,$

(b) $\dfrac{d}{dx} \int_a^x G(t)dt = G(x).$

Theorem 2 (Fundamental Theorem of Calculus: Indefinite Integral)
Let $G(x)$ be continuous and let $y(x)$ be continuously differentiable on $[a, b]$. Then for some constant c,

(a) $y(x) = \int \dfrac{dy}{dx}dx + c,$

(b) $\dfrac{d}{dx} \int G(x)dx = G(x).$

Part (a) of the fundamental theorem is used to find a candidate solution to a differential equation.

Part (b) of the fundamental theorem is used in differential equations to do an answer check.

The Method of Quadrature

The method is applied to differential equations $y' = f(x, y)$ in which f is independent of y. Then symbol y is absent from $f(x, y)$, which implies $f(x, y)$ is constant or else $f(x, y)$ depends only on the symbol x. The model differential equation then has the form $y' = F(x)$ where F is a given function of the single variable F.

(i) **To solve for $y(x)$ in $\dfrac{dy}{dx} = F(x)$, integrate on variable x across the equation, then use the Fundamental Theorem of Calculus.**

(ii) **Check the answer.**

Indefinite Integral Shortcut. Integrate across the equation with indefinite integrals, then collect all integration constants into symbol c.

Solution with Symbol c. Symbol c initially appears in the expression obtained for y. If no initial condition was given, then the answer for y is this expression, which contains the unresolved symbol c. Experts call this expression the **general solution**.

Solution with No symbol c. If an initial condition is given in the form $y = y_0$ at $x = x_0$ (same as $y(x_0) = y_0$), then symbol c can be resolved. For instance, if the answer is $y = 2(x - 1) + c$ and the initial condition is $y(-1) = 3$, then $y = 2(x - 1) + c$ with $x = -1, y = 3$ becomes $3 = 2(-1 - 1) + c$, and then $c = 7$. Experts call this expression a **particular solution**, meaning the symbol c has been resolved.

Theorem 3 (Existence-Uniqueness for Quadrature Equations)
Let $F(x)$ be continuous on $a < x < b$. Assume $a < x_0 < b$ and $-\infty < y_0 < \infty$. Then the initial value problem

(1) $$y' = F(x), \quad y(x_0) = y_0$$

has on interval $a < x < b$ the unique solution

(2) $$y(x) = y_0 + \int_{x_0}^{x} F(t)dt.$$

Details of proof appear on page 79.

Examples

1 Example (Quadrature) Solve $y' = 3e^x$, $y(0) = 0$.

Solution:
The final answer is $y = 3e^x - 3$. An answer check appears in the next example.
Details. The *shortcut* is applied.

$\frac{dy}{dx} = 3e^x$	Copy the differential equation.
$\int \frac{dy}{dx} dx = \int 3e^x dx$	Integrate across the equation on x.
$y(x) + c_1 = \int 3e^x dx$	Fundamental theorem of calculus, page 75.
$y(x) + c_1 = 3e^x + c_2$	Integral table.
$y(x) = 3e^x + c$	Where $c = c_2 - c_1$ is a constant.

The answer is $y = 3e^x + c$. The symbol c is to be resolved from the **initial condition** $y(0) = 0$, as follows.

$0 = y(0)$	Copy the initial condition (sides reversed).
$\quad = (3e^x + c)\|_{x=0}$	Insert $y = 3e^x + c$, the proposed solution.
$\quad = 3e^0 + c$	Substitute $x = 0$.

$$\begin{aligned} &= 3 + c && \text{Use } e^0 = 1. \\ c &= -1 && \text{Equation } 0 = 3 + c \text{ solved for } c. \end{aligned}$$

Candidate solution. Back-substitute the symbol c answer $c = -1$ into the answer $y = 3e^x + c$ to obtain the candidate solution $y = 3e^x + (-3)$. This answer can contain errors, in general, due to integration and arithmetic mistakes.

2 Example (Answer Check) Given $y' = 3e^x$, $y(0) = 0$ and candidate solution $y(x) = 3e^x - 3$, display an answer check.

Solution: There are **two panels** in this answer check: **Panel 1**: differential equation check, **Panel 2**: initial condition check.

Panel 1. We check the answer $y = 3e^x - 3$ for the differential equation $y' = 3e^x$.

The steps are:

$$\begin{aligned} \text{LHS} &= y' && \text{Left side of the differential equation.} \\ &= (3e^x - 3)' && \text{Substitute the expression for } y. \\ &= 3e^x - 0 && \text{Sum rule, constant rule and } (e^u)' = u'e^u. \\ &= \text{RHS} && \text{Solution verified.} \end{aligned}$$

Panel 2. We check the answer $y = 3e^x - 3$ against the initial condition $y(0) = 0$. Expected is an immediate mental check that $e^0 = 1$ implies the correctness of $y(0) = 0$.

The steps will be shown in order to detail the algorithm for checking an initial condition. The algorithm applies when checking complex algebraic expressions. Abbreviated versions of the algorithm are used on simple expressions.

$$\begin{aligned} \text{LHS} &= y(0) && \text{Left side of the initial condition } y(0) = 0. \\ &= (3e^x - 3)\big|_{x=0} && \text{Notation } y(x_0) \text{ means substitute } x = x_0 \\ & && \text{into the expression for } y. \\ &= 3e^0 - 3 && \text{Substitute } x = 0 \text{ into the expression.} \\ &= 0 && \text{Because } e^0 = 1. \\ &= \text{RHS} && \text{Initial condition verified.} \end{aligned}$$

River Crossing

A boat crosses a river at fixed speed with power applied perpendicular to the shoreline. Is it possible to estimate the boat's downstream location?

The answer is *yes*. The problem's variables are

x	Distance from shore,	w	Width of the river,
y	Distance downstream,	v_b	Boat velocity (dx/dt),
t	Time in hours,	v_r	River velocity (dy/dt).

The calculus chain rule $dy/dx = (dy/dt)/(dx/dt)$ is applied, using the symbols v_r and v_b instead of dy/dt and dx/dt, to give the *model equation*

$$(3) \qquad \frac{dy}{dx} = \frac{v_r}{v_b}.$$

Stream Velocity. The downstream river velocity will be approximated by $v_r = kx(w - x)$, where $k > 0$ is a constant. This equation gives velocity $v_r = 0$ at the two shores $x = 0$ and $x = w$, while the **maximum stream velocity** at the center $x = w/2$ is (see page 79)

$$(4) \qquad v_c = \frac{kw^2}{4}.$$

Special River-Crossing Model. The model equation (3) using $v_r = kx(w-x)$ and the constant k defined by (4) give the initial value problem

$$(5) \qquad \frac{dy}{dx} = \frac{4v_c}{v_b w^2} x(w - x), \quad y(0) = 0.$$

The solution of (5) by the method of quadrature is

$$(6) \qquad y = \frac{4v_c}{v_b w^2} \left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2 \right),$$

where w is the river's width, v_c is the river's midstream velocity and v_b is the boat's velocity. In particular, the boat's **downstream drift** on the opposite shore is $\frac{2}{3}w(v_c/v_b)$. See *Technical Details* page 79.

3 Example (River Crossing) A boat crosses a mile-wide river at 3 miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is 10 miles per hour. Find the transit time and the downstream drift to the opposite shore.

Solution: The answers, justified below, are 20 minutes and 20/9 miles.

Transit time. This is the time it takes to reach the opposite shore. The layman answer of 20 minutes is correct, because the boat goes 3 miles in one hour, hence 1 mile in 1/3 of an hour, perpendicular to the shoreline.

Downstream drift. This is the value $y(1)$, where y is the solution of equation (5), with $v_c = 10$, $v_b = 3$, $w = 1$, all distances in miles. The special model is

$$\frac{dy}{dx} = \frac{40}{3}x(1 - x), \quad y(0) = 0.$$

The solution given by equation (6) is $y = \frac{40}{3}\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$ and the downstream drift is then $y(1) = 20/9$ miles. This answer is 2/3 of the layman's answer of $(1/3)(10)$ miles. The explanation is that the boat is pushed downstream at a variable rate from 0 to 10 miles per hour, depending on its position x.

Details and Proofs

Proof of Theorem 3:

Uniqueness. Let $y(x)$ be any solution of (1). It will be shown that $y(x)$ is given by the solution formula (2).

$$y(x) = y(0) + \int_{x_0}^{x} y'(t)dt \qquad \text{Fundamental theorem of calculus, page 842.}$$
$$= y_0 + \int_{x_0}^{x} F(t)dt \qquad \text{Use (1). This verifies equation (2).}$$

Answer Check. Let $y(x)$ be given by solution formula (2). It will be shown that $y(x)$ solves initial value problem (1).

$$y'(x) = \left(y_0 + \int_{x_0}^{x} F(t)dt\right)' \qquad \text{Compute the derivative from (2).}$$
$$= F(x) \qquad \text{Apply the fundamental theorem of calculus.}$$

The initial condition is verified in a similar manner:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t)dt \qquad \text{Apply (2) with } x = x_0.$$
$$= y_0 \qquad \text{The integral is zero: } \int_a^a F(x)dx = 0.$$

The proof is complete.

Technical Details for (4): The maximum of a continuously differentiable function $f(x)$ on $0 \leq x \leq w$ can be found by locating the critical points (i.e., where $f'(x) = 0$) and then testing also the endpoints $x = 0$ and $x = w$. The derivative $f'(x) = k(w - 2x)$ is zero at $x = w/2$. Then $f(w/2) = kw^2/4$. This value is the maximum of f, because $f = 0$ at the endpoints.

Technical Details for (6): Let $a = \dfrac{4v_c}{v_b w^2}$. Then

$$y = y(0) + \int_0^x y'(t)dt \qquad \text{Method of quadrature.}$$
$$= 0 + a \int_0^x t(w - t)dt \qquad \text{By (5), } y' = at(w - t).$$
$$= a\left(-\tfrac{1}{3}x^3 + \tfrac{1}{2}wx^2\right). \qquad \text{Integral table.}$$

To compute the downstream drift, evaluate $y(w) = a\dfrac{w^3}{6}$ or $y(w) = \dfrac{2w}{3}\dfrac{v_c}{v_b}$.

Exercises 2.1

Quadrature. Find a candidate solution for each initial value problem and verify the solution. See Example 1 and Example 2, page 76.

1. $y' = 4e^{2x}$, $y(0) = 0$.

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1 + x)y' = x$, $y(0) = 0$.

4. $(1 - x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1 + x^2)y' = 1$, $y(0) = 0$.

12. $(1 + 4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1 + x)y' = 1$, $y(0) = 0$.

16. $(2 + x)y' = 2$, $y(0) = 0$.

17. $(2 + x)(1 + x)y' = 2$, $y(0) = 0$.

18. $(2 + x)(3 + x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

20. $y' = (1 + \cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing. A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 3, page 78, and the details for (6).

21. $w = 1$, $v_b = 4$, $v_c = 12$

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I. Verify the identity. Use the fundamental theorem of calculus part (b), page 75.

29. $\int_0^x (1 + t)^3 dt = \frac{1}{4}\left((1 + x)^4 - 1\right)$.

30. $\int_0^x (1 + t)^4 dt = \frac{1}{5}\left((1 + x)^5 - 1\right)$.

31. $\int_0^x te^{-t}dt = -xe^{-x} - e^{-x} + 1$.

32. $\int_0^x te^t dt = xe^x - e^x + 1$.

Fundamental Theorem II. Differentiate. Use the fundamental theorem of calculus part (b), page 75.

33. $\int_0^{2x} t^2 \tan(t^3)dt$.

34. $\int_0^{3x} t^3 \tan(t^2)dt$.

35. $\int_0^{\sin x} te^{t+t^2} dt$.

36. $\int_0^{\sin x} \ln(1 + t^3)dt$.

Fundamental Theorem III. Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page 75. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x - 1)$

38. $f(x) = x^2(x + 1)$

39. $f(x) = \cos(3\pi x/4)$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1 + x^2}$

42. $f(x) = \dfrac{2x}{1 + x^4})$

43. $f(x) = x^2 e^{x^3}$

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1 + x^2}}$

46. $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$

48. $f(x) = \dfrac{1}{\sqrt{1 + 4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1 + x^2}}$

50. $f(x) = \dfrac{4x}{\sqrt{1 - 4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1 + e^x}$

54. $f(x) = \dfrac{\ln|x|}{x}$

55. $f(x) = \sec^2 x$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x x$

60. $f(x) = \sec x \tan x x$

Integration by Parts. Integrate $\int_0^1 f(x)dx$ by parts, $\int u\,dv = uv - \int v\,du$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln|x|$

64. $f(x) = x \ln|x|$

65. $f(x) = x^2 e^{2x}$

66. $f(x) = (1 + 2x)e^{2x}$

67. $f(x) = x \cosh x$

68. $f(x) = x \sinh x$

69. $f(x) = x \arctan(x)$

70. $f(x) = x \arcsin(x)$

Partial Fractions. Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x + 4}{x + 5}$

72. $f(x) = \dfrac{x - 2}{x - 4}$

73. $f(x) = \dfrac{x^2 + 4}{(x + 1)(x + 2)}$

74. $f(x) = \dfrac{x(x - 1)}{(x + 1)(x + 2)}$

75. $f(x) = \dfrac{x + 4}{(x + 1)(x + 2)}$

76. $f(x) = \dfrac{x - 1}{(x + 1)(x + 2))}$

77. $f(x) = \dfrac{x + 4}{(x + 1)(x + 2)(x + 5)}$

78. $f(x) = \dfrac{x(x - 1)}{(x + 1)(x + 2)(x + 3)}$

79. $f(x) = \dfrac{x + 4}{(x + 1)(x + 2)(x - 1)}$

80. $f(x) = \dfrac{x(x - 1)}{(x + 1)(x + 2)(x - 1)}$

Special Methods. Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2 + 2}{(x + 1)^2}$, $u = x + 1$.

82. $f(x) = \dfrac{x^2 + 2}{(x - 1)^2}$, $u = x - 1$.

83. $f(x) = \dfrac{2x}{(x^2 + 1)^3}$, $u = x^2 + 1$.

84. $f(x) = \dfrac{3x^2}{(x^3 + 1)^2}$, $u = x^3 + 1$.

85. $f(x) = \dfrac{x^3 + 1}{x^2 + 1}$, use long division.

86. $f(x) = \dfrac{x^4 + 2}{x^2 + 1}$, use long division.