Chapter 4

First Order Numerical Methods

Contents

4.1 Solving $y' = F(x)$ Numerically

Studied here is the creation of numerical tables and graphics for the solution of the initial value problem

$$y' = F(x), \quad y(x_0) = y_0. \tag{1}$$

To illustrate, consider the initial value problem

$$y' = 3x^2 - 1, \quad y(0) = 2.$$

Quadrature gives the explicit **symbolic solution**

$$y(x) = x^3 - x + 2.$$

In Figure 1, evaluation of $y(x)$ from $x = 0$ to $x = 1$ in increments of 0.1 gives the xy-table, whose entries represent the **dots** for the **connect-the-dots** graphic.

x	y
0.0	2.000
0.1	1.901
0.2	1.808
0.3	1.727
0.4	1.664
0.5	1.625

x	y
0.6	1.616
0.7	1.643
0.8	1.712
0.9	1.829
1.0	2.000

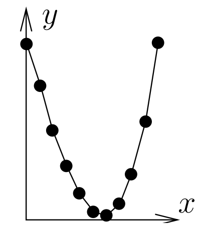


Figure 1. A table of xy-values for $y = x^3 - x + 2$.
The graphic represents the table's rows as *dots*, which are joined to make the *connect-the-dots* graphic.

The interesting case is when quadrature in (1) encounters an integral $\int_{x_0}^{x} F(t)dt$ that cannot be evaluated to provide an explicit symbolic equation for $y(x)$. Nevertheless, $y(x)$ can be computed numerically.

Applied here are numerical integration rules from calculus: *rectangular*, *trapezoidal* and *Simpson*; see page 230 for a review of the three rules. The ideas lead to the numerical methods of Euler, Heun and Runge-Kutta, which appear later in this chapter.

How to make an xy-table. Given $y' = F(x)$, $y(x_0) = y_0$, a table of xy-values is created as follows. The x-values are equally spaced a distance $h > 0$ apart. Each x, y pair in the table represents a *dot* in the *connect-the-dots* graphic of the explicit solution

$$y(x) = y_0 + \int_{x_0}^{x} F(t)dt.$$

First table entry. The *initial condition* $y(x_0) = y_0$ identifies two constants x_0, y_0 to be used for the first table pair X, Y. For example, $y(0) = 2$ identifies first table pair $X = 0$, $Y = 2$.

Second table entry. The second table pair X, Y is computed from the first table pair x_0, y_0 and a **recurrence**. The X-value is given by $X = x_0 + h$, while the Y-value is given by the numerical integration method being used, in accordance with Table 1 (the table is justified on page 233).

Table 1. Three numerical integration methods.

Rectangular Rule	$Y = y_0 + hF(x_0)$
Trapezoidal Rule	$Y = y_0 + \dfrac{h}{2}(F(x_0) + F(x_0 + h))$
Simpson's Rule	$Y = y_0 + \dfrac{h}{6}(F(x_0) + 4F(x_0 + h/2) + F(x_0 + h)))$

Third and higher table entries. They are computed by letting x_0, y_0 be the current table entry, then the next table entry X, Y is found exactly as outlined above for the second table entry.

It is expected, and normal, to compute the table entries using computer assist. In simple cases, a calculator will suffice. If F is complicated or Simpson's rule is used, then a computer algebra system or a numerical laboratory is recommended. See Example 2, page 227.

How to make a connect-the-dots graphic. To illustrate, consider the xy-pairs below, which are to represent the *dots* in the *connect-the-dots* graphic.

$$(0.0, 2.000), (0.1, 1.901), (0.2, 1.808), (0.3, 1.727), (0.4, 1.664),$$
$$(0.5, 1.625), (0.6, 1.616), (0.7, 1.643), (0.8, 1.712), (0.9, 1.829),$$
$$(1.0, 2.000).$$

Hand drawing. The method, unchanged from high school mathematics courses, is to plot the points as dots on an xy-coordinate system, then connect the dots with line segments. See Figure 2.

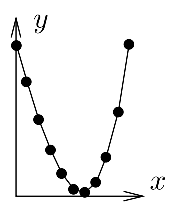


Figure 2. A Connect-the-Dots Graphic.
A computer-generated graphic made to simulate hand-drawn.

Computer algebra system graphic. The computer algebra system `maple` has a primitive syntax especially made for connect-the-dots graphics. Below, `Dots` is a list of xy-pairs.

```
Dots:=[0.0, 2.000], [0.1, 1.901], [0.2, 1.808],
      [0.3, 1.727], [0.4, 1.664], [0.5, 1.625],
      [0.6, 1.616], [0.7, 1.643], [0.8, 1.712],
      [0.9, 1.829], [1.0, 2.000]:
plot([Dots]);
```

The plotting of *points only* can be accomplished by adding options into the `plot` command: `type=point` and `symbol=circle` will suffice.

Numerical laboratory graphic. The computer programs `matlab`, `octave` and `scilab` provide primitive plotting facilities, as follows.

```
X=[0,.1,.2,.3,.4,.5,.6,.7,.8,.9,1]
Y=[2.000, 1.901, 1.808, 1.727, 1.664, 1.625,
   1.616, 1.643, 1.712, 1.829, 2.000]
plot(X,Y)
```

1 **Example (Rectangular Rule)** Consider $y' = 3x^2 - 2x$, $y(0) = 0$. Apply the rectangular rule to make an xy-table for $y(x)$ from $x = 0$ to $x = 2$ in steps of $h = 0.2$. Graph the approximate solution and the exact solution $y(x) = x^3 - x^2$ for $0 \le x \le 2$.

Solution: The exact solution $y = x^3 - x^2$ is verified directly, by differentiation. It was obtained by quadrature applied to $y' = 3x^2 - 2x$, $y(0) = 0$.

The first table entry 0, 0 is used to obtain the second table entry $X = 0.2$, $Y = 0$ as follows.

$x_0 = 0,\ y_0 = 0$	The current table entry, row 1.
$X = x_0 + h$	The next table entry, row 2.
$\quad = 0.2,$	Use $x_0 = 0,\ h = 0.2$.
$Y = y_0 + hF(x_0)$	Rectangular rule.
$\quad = 0 + 0.2(0).$	Use $h = 0.2,\ F(x) = 3x^2 - 2x$.

The remaining 9 rows of the table are completed by calculator, following the pattern above for the second table entry. The result:

Table 2. Rectangular rule solution and exact values for $y' = 3x^2 - 2x$, $y(0) = 0$ on $0 \le x \le 2$, step size $h = 0.2$.

x	y-rect	y-exact	x	y-rect	y-exact
0.0	0.000	0.000	1.2	0.120	0.288
0.2	0.000	−0.032	1.4	0.504	0.784
0.4	−0.056	−0.096	1.6	1.120	1.536
0.6	−0.120	−0.144	1.8	2.016	2.592
0.8	−0.144	−0.128	2.0	3.240	4.000
1.0	−0.080	0.000			

The xy-values from the table are used to obtain the comparison plot in Figure 3.

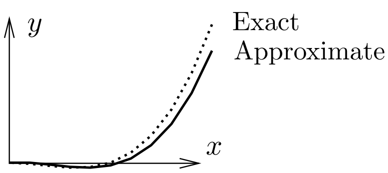


Figure 3. Comparison Plot. Rectangular rule numerical solution and the exact solution for $y = x^3 - x^2$ for $y' = 3x^2 - 2x$, $y(0) = 0$.

2 Example (Trapezoidal Rule) Consider $y' = \cos x + 2x$, $y(0) = 0$. Apply both the rectangular and trapezoidal rules to make an xy-table for $y(x)$ from $x = 0$ to $x = \pi$ in steps of $h = \pi/10$. Compare the two approximations in a graphic for $0 \le x \le \pi$.

Solution: The exact solution $y = \sin x + x^2$ is verified directly, by differentiation. It will be seen that the trapezoidal solution is nearly identical, graphically, to the exact solution.

The table will have 11 rows. The three columns are x, y-rectangular and y-trapezoidal. The first table entry 0, 0, 0 is used to obtain the second table entry 0.1π, 0.31415927, 0.40516728 as follows.

Rectangular rule second entry.

$Y = y_0 + hF(x_0)$	Rectangular rule.
$\quad = 0 + h(\cos 0 + 2(0))$	Use $F(x) = \cos x + 2x,\ x_0 = y_0 = 0$.

$= 0.31415927.$ Use $h = 0.1\pi = 0.31415927.$

Trapezoidal rule second entry.

$$\begin{aligned} Y &= y_0 + 0.5h(F(x_0) + F(x_0 + h)) && \text{Trapezoidal rule.} \\ &= 0 + 0.05\pi(\cos 0 + \cos h + 2h) && \text{Use } x_0 = y_0 = 0,\ F(x) = \cos x + 2x. \\ &= 0.40516728. && \text{Use } h = 0.1\pi. \end{aligned}$$

The remaining 9 rows of the table are completed by calculator, following the pattern above for the second table entry. The result:

Table 3. Rectangular and trapezoidal solutions for $y' = \cos x + 2x$, $y(0) = 0$ on $0 \le x \le \pi$, step size $h = 0.1\pi$.

x	y-rect	y-trap	x	y-rect	y-trap
0.000000	0.000000	0.000000	1.884956	4.109723	4.496279
0.314159	0.314159	0.405167	2.199115	5.196995	5.638458
0.628319	0.810335	0.977727	2.513274	6.394081	6.899490
0.942478	1.459279	1.690617	2.827433	7.719058	8.300851
1.256637	2.236113	2.522358	3.141593	9.196803	9.869604
1.570796	3.122762	3.459163			

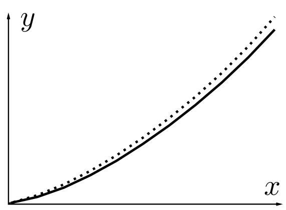


Figure 4. Comparison Plot.
Rectangular (solid) and trapezoidal (dotted) numerical solutions for $y' = \cos x + 2x$, $y(0) = 0$ for $h = 0.1\pi$ on $0 \le x \le \pi$.

Computer algebra system. The `maple` implementation for Example 2 appears below. The code produces lists `Dots1` and `Dots2` which contain Rectangular (left panel) and Trapezoidal (right panel) approximations.

```
# Rectangular algorithm
# Group 1, initialize.
F:=x->evalf(cos(x) + 2*x):
x0:=0:y0:=0:h:=0.1*Pi:
Dots1:=[x0,y0]:

# Group 2, loop count = 10
for i from 1 to 10 do
Y:=y0+h*F(x0):
x0:=x0+h:y0:=evalf(Y):
Dots1:=Dots1,[x0,y0];
end do;

# Group 3, plot.
plot([Dots1]);
```

```
# Trapezoidal algorithm
# Group 1, initialize.
F:=x->evalf(cos(x) + 2*x):
x0:=0:y0:=0:h:=0.1*Pi:
Dots2:=[x0,y0]:

# Group 2, repeat 10 times
for i from 1 to 10 do
Y:=y0+h*(F(x0)+F(x0+h))/2:
x0:=x0+h:y0:=evalf(Y):
Dots2:=Dots2,[x0,y0];
end do;

# Group 3, plot.
plot([Dots2]);
```

3 Example (Simpson's Rule) Consider $y' = e^{-x^2}$, $y(0) = 0$. Apply both the rectangular and Simpson rules to make an xy-table for $y(x)$ from $x = 0$ to $x = 1$ in steps of $h = 0.1$. In the table, include values for the exact solution $y(x) = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. Compare the two approximations in a graphic for $0.8 \le x \le 1.0$.

Solution: The **error function** $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is a library function available in `maple`, `mathematica`, `matlab` and other computing platforms. It is known that the integral cannot be expressed in terms of elementary functions.

The xy-table. There will be 11 rows, for $x = 0$ to $x = 1$ in steps of $h = 0.1$. There are four columns: x, y-rectangular, y-Simpson, y-exact.

The first row arises from $y(0) = 0$, giving the four entries 0, 0, 0, 0. It will be shown how to obtain the second row by calculator methods, for the two algorithms *rectangular* and *Simpson*.

Rectangular rule second entry.

$Y1 = y_0 + hF(x_0)$	Rectangular rule.
$= 0 + h(e^0)$	Use $F(x) = e^{-x^2}$, $x_0 = y_0 = 0$.
$= 0.1.$	Use $h = 0.1$.

Simpson rule second entry.

$Y2 = y_0 + \frac{h}{6}(F(x_0) + 4F(x_1) + F(x_2))$	Simpson rule, $x_1 = x_0 + h/2$, $x_2 = x_0 + h$.
$= 0 + \frac{h}{6}(e^0 + 4e^{.5} + e^{.1})$	Use $F(x) = e^{-x^2}$, $x_0 = y_0 = 0$.
$= 0.09966770540.$	Use $h = 0.1$.

Exact solution second entry.
The numerical work requires the tabulated function $\operatorname{erf}(x)$. The `maple` details:

```
x0:=0:y0:=0:h:=0.1:            Given.
c:=sqrt(Pi)/2                  Conversion factor.
Exact:=x->y0+c*erf(x):         Exact solution y = y0 + int_0^x e^(-t^2) dt.
Y3:=Exact(x0+h);               Calculate exact answer.
# Y3 := .09966766428
```

Table 4. Rectangular and Simpson Rule.
Numerical solutions for $y' = e^{-x^2}$, $y(0) = 0$ on $0 \le x \le \pi$, step size $h = 0.1$.

x	y-rect	y-Simp	y-exact
0.0	0.00000000	0.00000000	0.00000000
0.1	0.10000000	0.09966771	0.09966766
0.2	0.19900498	0.19736511	0.19736503
0.3	0.29508393	0.29123799	0.29123788
0.4	0.38647705	0.37965297	0.37965284
0.5	0.47169142	0.46128114	0.46128101
0.6	0.54957150	0.53515366	0.53515353
0.7	0.61933914	0.60068579	0.60068567
0.8	0.68060178	0.65766996	0.65766986
0.9	0.73333102	0.70624159	0.70624152
1.0	0.77781682	0.74682418	0.74682413

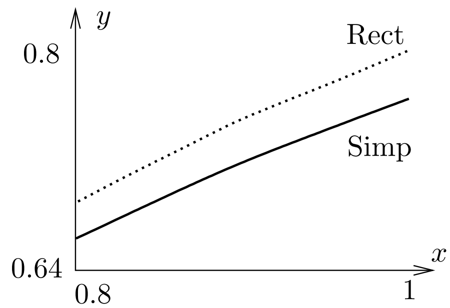


Figure 5. Comparison Plot.

Rectangular (dotted) and Simpson (solid) numerical solutions for $y' = e^{-x^2}$, $y(0) = 0$ for $h = 0.1$ on $0.8 \le x \le 1.0$.

Computer algebra system. The `maple` implementation for Example 3 appears below. The code produces two lists `Dots1` and `Dots2` which contain Rectangular (left panel) and Simpson (right panel) approximations.

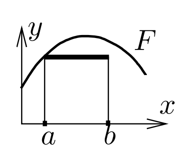
```
# Rectangular algorithm
# Group 1, initialize.
F:=x->evalf(exp(-x*x)):
x0:=0:y0:=0:h:=0.1:
Dots1:=[x0,y0]:

# Group 2, repeat 10 times
for i from 1 to 10 do
Y:=evalf(y0+h*F(x0)):
x0:=x0+h:y0:=Y:
Dots1:=Dots1,[x0,y0];
end do;


# Group 3, plot.
plot([Dots1]);
```

```
# Simpson algorithm
# Group 1, initialize.
F:=x->evalf(exp(-x*x)):
x0:=0:y0:=0:h:=0.1:
Dots2:=[x0,y0]:

# Group 2, loop count = 10
for i from 1 to 10 do
Y:=evalf(y0+h*(F(x0)+
   4*F(x0+h/2)+F(x0+h))/6):
x0:=x0+h:y0:=Y:
Dots2:=Dots2,[x0,y0];
end do;

# Group 3, plot.
plot([Dots2]);
```

Review of Numerical Integration

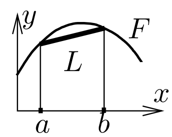
Reproduced here are calculus topics: the **rectangular rule**, the **trapezoidal rule** and **Simpson's rule** for the numerical approximation of an integral $\int_a^b F(x)dx$. The approximations are valid for $b - a$ small. Larger intervals must be subdivided, then the rule applies to the small subdivisions.

Rectangular Rule. The approximation uses Euler's idea of replacing the integrand by a constant. The value of the integral is approximately the area of a rectangle of width $b - a$ and height $F(a)$.



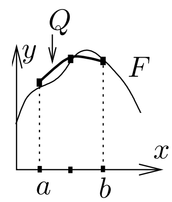
(2) $$\int_a^b F(x)dx \approx (b - a)F(a).$$

Trapezoidal Rule. The rule replaces the integrand $F(x)$ by a linear function $L(x)$ which connects the planar points $(a, F(a))$, $(b, F(b))$. The value of the integral is approximately the area under the curve L, which is the area of a trapezoid.



(3) $$\int_a^b F(x)dx \approx \frac{b-a}{2}\left(F(a) + F(b)\right).$$

Simpson's Rule. The rule replaces the integrand $F(x)$ by a quadratic polynomial $Q(x)$ which connects the planar points $(a, F(a))$, $((a+b)/2, F((a+b)/2))$, $(b, F(b))$. The value of the integral is approximately the area under the quadratic curve Q.



(4) $$\int_a^b F(x)dx \approx \frac{b-a}{6}\left(F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right).$$

Simpson's Polynomial Rule. If $Q(x)$ is constant, or a linear, quadratic or cubic polynomial, then (proof on page 232)

(5) $$\int_a^b Q(x)dx = \frac{b-a}{6}\left(Q(a) + 4Q\left(\frac{a+b}{2}\right) + Q(b)\right).$$

Integrals of linear, quadratic and cubic polynomials can be evaluated *exactly* using Simpson's polynomial rule (5); see Example 4, page 231.

Remarks on Simpson's Rule. The right side of (4) is exactly the integral of $Q(x)$, which is evaluated by equation (5). The appearance of F instead of Q on the right in equation (4) is due to the relations $Q(a) = F(a)$, $Q((a+b)/2) = F((a+b)/2)$, $Q(b) = F(b)$, which arise from the requirement that Q connect three points along curve F.

The quadratic interpolation polynomial $Q(x)$ is determined uniquely from the three data points; see *Quadratic Interpolant*, page 232, for a formula for Q and a derivation. It is interesting that Simpson's rule depends only upon the uniqueness and not upon the actual formula for Q!

4 **Example (Polynomial Quadrature)** Apply Simpson's polynomial rule (5) to verify $\int_1^2 (x^3 - 16x^2 + 4)dx = -355/12$.

Solution: The application proceeds as follows:

$I = \int_1^2 Q(x)dx$	Evaluate integral I using $Q(x) = x^3 - 16x^2 + 4$.

$$= \frac{2-1}{6}\left(Q(1)+4Q(3/2)+Q(2)\right)$$ Apply Simpson's polynomial rule (5).

$$= \frac{1}{6}\left(-11+4(-229/8)-52\right)$$ Use $Q(x)=x^3-16x^2+4$.

$$= -\frac{355}{12}.$$ Equality verified.

Simpson's Polynomial Rule Proof. Let $Q(x)$ be a linear, quadratic or cubic polynomial. It will be verified that

$$\int_a^b Q(x)dx = \frac{b-a}{6}\left(Q(a)+4Q\left(\frac{a+b}{2}\right)+Q(b)\right). \tag{6}$$

If the formula holds for polynomial Q and c is a constant, then the formula also holds for the polynomial cQ. Similarly, if the formula holds for polynomials Q_1 and Q_2, then it also holds for Q_1+Q_2. Consequently, it suffices to show that the formula is true for the special polynomials 1, x, x^2 and x^3, because then it holds for all combinations $Q(x)=c_0+c_1x+c_2x^2+c_3x^3$.

Only the special case $Q(x)=x^3$ will be treated here. The other cases are left to the exercises. The details:

$$\text{RHS} = \frac{b-a}{6}\left(Q(a)+4Q\left(\frac{a+b}{2}\right)+Q(b)\right)$$ Evaluate the right side of equation (6).

$$= \frac{b-a}{6}\left(a^3+\frac{1}{2}(a+b)^3+b^3\right)$$ Substitute $Q(x)=x^3$.

$$= \frac{b-a}{6}\left(\frac{3}{2}\right)\left(a^3+a^2b+ab^2+b^3\right)$$ Expand $(a+b)^3$. Simplify.

$$= \frac{1}{4}\left(b^4-a^4\right),$$ Multiply and simplify.

$$\text{LHS} = \int_a^b Q(x)dx$$ Evaluate the left hand side (LHS) of equation (6).

$$= \int_a^b x^3dx$$ Substitute $Q(x)=x^3$.

$$= \frac{1}{4}\left(b^4-a^4\right)$$ Evaluate.

$= \text{RHS}$. Compare with the RHS.

This completes the proof of Simpson's polynomial rule.

Quadratic Interpolant Q.

Given $a<b$ and the three data points (a,Y_0), $((a+b)/2,Y_1))$, $(b,Y_2))$, then there is a unique quadratic curve $Q(X)$ which connects the points, given by

$$\begin{aligned} Q(X) &= Y_0 + (4Y_1 - Y_2 - 3Y_0)\frac{X-a}{b-a} \\ &\quad + (2Y_2 + 2Y_0 - 4Y_1)\frac{(X-a)^2}{(b-a)^2}. \end{aligned} \tag{7}$$

Proof: The term *quadratic* is meant loosely: it can be a constant or linear function as well.

Uniqueness of the interpolant Q is established by subtracting two candidates to obtain a polynomial P of degree at most two which vanishes at three distinct points. By Rolle's theorem, P' vanishes at two distinct points and hence P'' vanishes at one point. Writing $P(X) = c_0 + c_1X + c_2X^2$ shows $c_2 = 0$ and then $c_1 = c_0 = 0$, or briefly, $P \equiv 0$. Hence the two candidates are identical.

It remains to verify the given formula (7). The details are presented as two lemmas.[1] The first lemma contains the essential ideas. The second simply translates the variables.

Lemma 1 Given y_1 and y_2, define $A = y_2 - y_1$, $B = 2y_1 - y_2$. Then the quadratic $y = x(Ax + B)$ fits the data items $(0, 0)$, $(1, y_1)$, $(2, 2y_2)$.

Lemma 2 Given Y_0, Y_1 and Y_2, define $y_1 = Y_1 - Y_0$, $y_2 = \frac{1}{2}(Y_2 - Y_0)$, $A = y_2 - y_1$, $B = 2y_1 - y_2$ and $x = 2(X - a)/(b - a)$. Then quadratic $Y(X) = Y_0 + x(Ax + B)$ fits the data items (a, Y_0), $((a + b)/2, Y_1)$, (b, Y_2).

To verify the first lemma, the formula $y = x(Ax + B)$ is tested to go through the given data points $(0, 0)$, $(1, y_1)$ and $(2, 2y_2)$. For example, the last pair is tested by the steps

$y(2) = 2(2A + B)$	Apply $y = x(Ax + B)$ with $x = 2$.
$= 4y_2 - 4y_1 + 4y_1 - 2y_2$	Use $A = y_2 - y_1$ and $B = 2y_1 - y_2$.
$= 2y_2$.	Therefore, the quadratic fits data item $(2, 2y_2)$.

The other two data items are tested similarly, details omitted here.

To verify the second lemma, observe that it is just a change of variables in the first lemma, $Y = Y_0 + y$. The data fit is checked as follows:

$Y(b) = Y_0 + y(2)$	Apply formulas $Y(X) = Y_0 + y(x)$, $y(x) = x(Ax + B)$ with $X = b$ and $x = 2$.
$= Y_0 + 2y_2$	Apply data fit $y(2) = 2y_2$.
$= Y_2$.	The quadratic fits the data item (b, Y_2).

The other two items are checked similarly, details omitted here. This completes the proof of the two lemmas. The formula for Q is obtained from the second lemma as $Q = Y_0 + Bx + Ax^2$ with substitutions for A, B and x performed to obtain the given equation for Q in terms of Y_0, Y_1, Y_2, a, b and X.

Justification of Table 1: The method of quadrature applied to $y' = F(x)$, $y(x_0) = y_0$ gives an explicit solution $y(x)$ involving the integral of F. Specialize this solution formula to $x = x_0 + h$ where $h > 0$. Then

$$y(x_0 + h) = y_0 + \int_{x_0}^{x_0+h} F(t)dt.$$

All three methods in Table 1 are derived by replacment of the integral above by the corresponding approximation taken from the rectangular, trapezoidal or

[1] What's a lemma? It's a helper theorem, used to dissect long proofs into short pieces.

Simpson method on page 230. For example, the trapezoidal method gives

$$\int_{x_0}^{x_0+h} F(t)dt \approx \frac{h}{2}\left(F(x_0) + F(x_0+h)\right),$$

whereupon replacement into the formula for y gives the entry in Table 1 as

$$Y \approx y(x_0+h) \approx y_0 + \frac{h}{2}\left(F(x_0) + F(x_0+h)\right).$$

This completes the justification of Table 1.

Exercises 4.1

Connect-the-Dots. Make a numerical table of 6 rows and a connect-the-dots graphic for the following.

1. $y = 2x + 5$, $x = 0$ to $x = 1$

2. $y = 3x + 5$, $x = 0$ to $x = 2$

3. $y = 2x^2 + 5$, $x = 0$ to $x = 1$

4. $y = 3x^2 + 5$, $x = 0$ to $x = 2$

5. $y = \sin x$, $x = 0$ to $x = \pi/2$

6. $y = \sin 2x$, $x = 0$ to $x = \pi/4$

7. $y = x \ln|1 + x|$, $x = 0$ to $x = 2$

8. $y = x \ln|1 + 2x|$, $x = 0$ to $x = 1$

9. $y = xe^x$, $x = 0$ to $x = 1$

10. $y = x^2e^x$, $x = 0$ to $x = 1/2$

Rectangular Rule. Apply the rectangular rule to make an xy-table for $y(x)$ with 11 rows and step size $h = 0.1$. Graph the approximate solution and the exact solution. Follow example 1.

11. $y' = 2x$, $y(0) = 5$.

12. $y' = 3x^2$, $y(0) = 5$.

13. $y' = 3x^2 + 2x$, $y(0) = 4$.

14. $y' = 3x^2 + 4x^3$, $y(0) = 4$.

15. $y' = \sin x$, $y(0) = 1$.

16. $y' = 2\sin 2x$, $y(0) = 1$.

17. $y' = \ln(1 + x)$, $y(0) = 1$. Exact $(1 + x)\ln|1 + x| + 1 - x$.

18. $y' = 2\ln(1 + 2x)$, $y(0) = 1$. Exact $(1 + 2x)\ln|1 + 2x| + 1 - 2x$.

19. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

20. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Trapezoidal Rule. Apply the trapezoidal rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 2.

21. $y' = 2x$, $y(0) = 1$.

22. $y' = 3x^2$, $y(0) = 1$.

23. $y' = 3x^2 + 2x$, $y(0) = 2$.

24. $y' = 3x^2 + 4x^3$, $y(0) = 2$.

25. $y' = \sin x$, $y(0) = 4$.

26. $y' = 2\sin 2x$, $y(0) = 4$.

27. $y' = \ln(1 + x)$, $y(0) = 1$. Exact $(1 + x)\ln|1 + x| + 1 - x$.

28. $y' = 2\ln(1 + 2x)$, $y(0) = 1$. Exact $(1 + 2x)\ln|1 + 2x| + 1 - 2x$.

29. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

30. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson Rule. Apply Simpson's rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 3.

31. $y' = 2x$, $y(0) = 2$.

32. $y' = 3x^2$, $y(0) = 2$.

33. $y' = 3x^2 + 2x$, $y(0) = 3$.

34. $y' = 3x^2 + 4x^3$, $y(0) = 3$.

35. $y' = \sin x$, $y(0) = 5$.

36. $y' = 2\sin 2x$, $y(0) = 5$.

37. $y' = \ln(1 + x)$, $y(0) = 1$. Exact $(1 + x)\ln|1 + x| + 1 - x$.

38. $y' = 2\ln(1 + 2x)$, $y(0) = 1$. Exact $(1 + 2x)\ln|1 + 2x| + 1 - 2x$.

39. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

40. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson's Rule. The following exercises use formulas and techniques found in the proof on page 232 and in Example 4, page 231.

41. Verify with Simpson's rule (5) for cubic polynomials the equality $\int_1^2 (x^3 + 16x^2 + 4)dx = 541/12$.

42. Verify with Simpson's rule (5) for cubic polynomials the equality $\int_1^2 (x^3 + x + 14)dx = 77/4$.

43. Let $f(x)$ satisfy $f(0) = 1$, $f(1/2) = 6/5$, $f(1) = 3/4$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 131/120$.

44. Let $f(x)$ satisfy $f(0) = -1$, $f(1/2) = 1$, $f(1) = 2$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 5/6$.

45. Verify Simpson's equality (5), assuming $Q(x) = 1$ and $Q(x) = x$.

46. Verify Simpson's equality (5), assuming $Q(x) = x^2$.

Quadratic Interpolation. The following exercises use formulas and techniques from the proof on page 232.

47. Verify directly that the quadratic polynomial $y = x(7 - 4x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -2)$.

48. Verify directly that the quadratic polynomial $y = x(8 - 5x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -4)$.

49. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, 1)$, $(0.5, 1.2)$, $(1, 0.75)$.

50. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, -1)$, $(0.5, 1)$, $(1, 2)$.

51. Verify the remaining cases in Lemma 1, page 233.

52. Verify the remaining cases in Lemma 2, page 233.