Laplace Ch10

Lerch's Theorem

If $\mathcal{L}(f) = \mathcal{L}(g)$ for $s \geq s_0$, Then $f(t) = g(t)$

<u>Example.</u> If $\mathcal{L}(y(t)) = \mathcal{L}(te^{-t})$, Then $y(t) = te^{-t}$
This is The basic cancellation law for solving equations.

Laplace Table

$f(t)$	$\int_0^\infty e^{-st} f(t)\,dt$	
1	$\frac{1}{s}$	
t^n	$\frac{n!}{s^{n+1}}$	
e^{at}	$\frac{1}{s-a}$	
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	
$H(t-a)$	e^{-as}/s	$(a \geq 0)$

Laplace Rules

$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$ (Direct transform)

$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$ (Linearity)

$\mathcal{L}((-t)f(t)) = \frac{d}{ds}[\mathcal{L}(f)]$ (s-diff)

$\mathcal{L}(f'(t)) = s\mathcal{L}(f) - f(0)$ (t-diff)

$\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f)\big|_{s \mapsto (s-a)}$ (shift)

$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f)$, $\mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a))$

Other rules:

- periodic rule
- convolution
- Integral

Evaluate $\mathcal{L}(5e^{1-t})$

$$\mathcal{L}(5e^{1-t}) = \mathcal{L}(5e\,e^{-t})$$
$$= 5e\,\mathcal{L}(e^{-t})$$
$$= 5e\,\frac{1}{s-(-1)}$$
$$= \boxed{\frac{5e}{s+1}}$$

- Exponential rule $e^a e^b = e^{a+b}$
- Linearity of $\mathcal{L}$
- $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

Evaluate $\mathcal{L}(e^{-t/3} + \sinh(-t/3))$

$$f(t) = e^{-t/3} + \sinh(-t/3)$$
$$= e^{-t/3} + \tfrac{1}{2}(e^{-t/3} - e^{t/3})$$
$$= \tfrac{3}{2}e^{-t/3} - \tfrac{1}{2}e^{t/3}$$

- $\sinh(u) \equiv \frac{1}{2}(e^u - e^{-u})$

$$\mathcal{L}(f) = \tfrac{3}{2}\mathcal{L}(e^{-t/3}) - \tfrac{1}{2}\mathcal{L}(e^{t/3})$$
$$= \frac{3}{2}\,\frac{1}{s-(-1/3)} - \frac{1}{2}\,\frac{1}{s-1/3}$$
$$= \boxed{\frac{1.5}{s+1/3} - \frac{0.5}{s-1/3}}$$

- Linearity of $\mathcal{L}$
- $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

Maple might return The answer as a single fraction, e.g.,

$$\frac{s - 2/3}{s^2 - 1/9}$$

Generally, transform answers are best left unsimplified and unchanged from The first instance of a valid expression. Only when checking answers does The issue of other forms of The same answer become an issue.

Given $f(t) = t^2\cos(3t)$, find $\mathcal{L}(f(t))$.

$$\mathcal{L}(f(t)) = \mathcal{L}\left((-t)^2\cos(3t)\right)$$

- prepare to use s-diff rule $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$.

$$= \frac{d}{ds}\frac{d}{ds}\mathcal{L}(\cos 3t)$$

- apply s-diff rule twice

$$= \left(\frac{d}{ds}\right)^2\left(\frac{s}{s^2+9}\right)$$

- Table

$$= \boxed{\frac{8s^3}{(s^2+9)^3} - \frac{6s}{(s^2+9)^2}}$$

- Found second derivative. Verified in Maple. See details at bottom of page.

Given $f(t) = \sin^2(3t)$, find $\mathcal{L}(f(t))$

$$\mathcal{L}(f(t)) = \mathcal{L}\left(\sin^2(3t)\right)$$

- Given

$$= \mathcal{L}\left(\frac{1}{2} - \frac{1}{2}\cos(6t)\right)$$

- use $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

$$= \mathcal{L}\left(\frac{1}{2}\right) - \mathcal{L}\left(\frac{1}{2}\cos(6t)\right)$$

- linearity

$$= \frac{1}{2}\mathcal{L}(1) - \frac{1}{2}\mathcal{L}(\cos 6t)$$

- linearity again

$$= \boxed{\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+36}}$$

- Tables

Details:

$$\frac{d}{ds}\left(\frac{s}{s^2+9}\right) = \frac{1}{s^2+9} - \frac{2s^2}{(s^2+9)^2}$$

- prod rule $(uv)' = u'v + uv'$ applied to $u = s$, $v = (s^2+9)^{-1}$.

$$\frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right) = \frac{d}{ds}\left(\frac{1}{s^2+9}\right) - \frac{d}{ds}\left(\frac{2s^2}{(s^2+9)^2}\right)$$

$$= \frac{-2s}{(s^2+9)^2} - \frac{4s}{(s^2+9)^2} + \frac{(2s^2)(2)(2s)}{(s^2+9)^3}$$

$$= \frac{8s^3}{(s^2+9)^3} - \frac{6s}{(s^2+9)^2}$$

Given $f(t) = e^t\sin(t) - e^{2t}\cos(4t)$, find $\mathcal{L}(f(t))$.

$$\mathcal{L}(e^t\sin t) = \mathcal{L}(\sin t)\Big|_{s\to s-1}$$

- Apply the shifting Th... see below

$$= \frac{1}{s^2+1}\Big|_{s\to s-1}$$

$$= \frac{1}{(s-1)^2+1}$$

$$\mathcal{L}(e^{2t}\cos(4t)) = \mathcal{L}(\cos 4t)\Big|_{s\to s-2}$$

- Apply the shifting Theo...

$$= \frac{s}{s^2+4^2}\Big|_{s\to s-2}$$

$$= \frac{s-2}{(s-2)^2+16}$$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^t\sin t) - \mathcal{L}(e^{2t}\cos 4t)$$

- linearity of $\mathcal{L}$

$$= \boxed{\frac{1}{(s-1)^2+1} - \frac{s-2}{(s-2)^2+16}}$$

<u>Shifting Theorem</u> $\mathcal{L}(e^{at}f(t))$ equals $\mathcal{L}(f(t))$ with s replaced by s... i.e.,

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\Big|_{s\to s-a}.$$

Proof:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

- Direct transform

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

- $e^a e^b = e^{a+b}$

$$= \int_0^\infty e^{-st}f(t)\,dt \quad \text{with } s \to s-a$$

$$= \mathcal{L}(f(t))\Big|_{s\to s-a}.$$

Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{3}{s^2} + \frac{s+1}{s^2+1}$

$$\mathcal{L}(f(t)) = \frac{3}{s^2} + \frac{s+1}{s^2+1}$$ • given

$$= 3\left(\frac{1}{s^2}\right) + (1)\left(\frac{s}{s^2+1}\right) + (1)\left(\frac{1}{s^2+1}\right)$$ • Arrange for Table usage

$$= 3\mathcal{L}(t) + (1)\mathcal{L}(\cos t) + (1)\mathcal{L}(\sin t)$$ • Use table

$$= \mathcal{L}(3t + \cos t + \sin t)$$ • Linearity of $\mathcal{L}$.

$$f(t) = \boxed{3t + \cos t + \sin t}$$ • Apply Lerch's cancellation law

Evaluate $\mathcal{L}(1.1 - (t-1)(t+1)(t-2))$

Let $f(t) = 1.1 - (t-1)(t+1)(t-2)$ • Given

$$= 1.1 - (t^2-1)(t-2)$$ • Multiply

$$= 1.1 - t^3 + 2t^2 + t - 2$$

$$= -t^3 + 2t^2 + t - 0.9$$

Then

$$\mathcal{L}(f(t)) = \mathcal{L}(-t^3 + 2t^2 + t - 0.9)$$

$$= -\mathcal{L}(t^3) + 2\mathcal{L}(t^2) + \mathcal{L}(t) - 0.9\mathcal{L}(1)$$ • Linearity

$$= -\frac{3!}{s^4} + 2\frac{2!}{s^3} + \frac{1}{s^2} - \frac{0.9}{s}$$ • Table

$$= \boxed{-\frac{6}{s^4} + \frac{4}{s^3} + \frac{1}{s^2} - \frac{9/10}{s}}$$

Solve for $f(t)$ in the equality $\mathcal{L}(f(t)) = \frac{s+1}{(s-1)(s-2)}$.

$$\mathcal{L}(f(t)) = \frac{s+1}{(s-1)(s+2)}$$ • Given

$$= \frac{A}{s-1} + \frac{B}{s+2}$$ • Theory of partial fractions

$$= A\mathcal{L}(e^t) + B\mathcal{L}(e^{-2t})$$ • $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$$= \mathcal{L}(Ae^t + Be^{-2t})$$ • Linearity of $\mathcal{L}$.

$$f(t) = Ae^t + Be^{-2t}$$ • Lerch's cancellation law

$$= \boxed{\left(\tfrac{2}{3}\right)e^t + \left(\tfrac{1}{3}\right)e^{-2t}}$$ • Solve partial fraction problem, $A = \frac{2}{3}$, $B = \frac{1}{3}$. See below.

Solve $\frac{s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$ for A, B

cross-multiply by $s-1$ and then set $s-1 = 0$:

$$\frac{s+1}{s+2} = A + \frac{B}{s+2}(s-1)$$

$$\frac{1+1}{1+2} = A + \frac{B}{1+2}(1-1)$$ • set $s-1=0$

$$= A$$

Then $\boxed{A = \frac{2}{3}}$. similarly, cross-multiply by $s+2$ and put $s+2=0$:

$$\frac{s+1}{s-1} = \frac{A}{s-1}(s+2) + B$$

$$\frac{-2+1}{-2-1} = \frac{A}{s-1}(-2+2) + B$$

Then $\boxed{B = \frac{1}{3}}$.

Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \arctan(1/s)$

The answer is $f(t) = \dfrac{\sin t}{t}$, obtained as follows.

$\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ • s-diff rule

$= \frac{d}{ds}\left(\arctan(1/s)\right)$

$= \dfrac{-s^{-2}}{1+(1/s)^2}$ • Do the differentiation

$= \dfrac{-1}{s^2+1}$

$= -\mathcal{L}(\sin t)$ • Table

$= \mathcal{L}(-\sin t)$ • Linearity of $\mathcal{L}$

$(-t)f(t) = -\sin t$ • Apply Lerch's cancellation law.

$f(t) = \boxed{\dfrac{\sin t}{t}}$ • Divide to find f.

s-diff rule. Multiplication of $f(t)$ by $(-t)$ differentiates the transform: $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$.

Proof:

$\mathcal{L}((-t)f(t)) = \int_0^\infty (-t)f(t)e^{-st}\,dt$ • Direct transform

$= \int_0^\infty f(t)(-t)e^{-st}\,dt$

$= \int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt$

$= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$ • Possible because of convergence properties of the integral.

$= \frac{d}{ds}\mathcal{L}(f(t))$.

Solve $\begin{cases} x'' + 3x' + 2x = \frac{1}{2} + e^{-3t} \\ x(0)=0,\quad x'(0)=0 \end{cases}$

By the Laplace method.

Step 1

$x'' + 3x' + 2x = \frac{1}{2} + e^{-3t}$ — Given DE

• $\mathcal{L}(x''+3x'+2x) = \mathcal{L}(\frac{1}{2}+e^{-3t})$ — Take Laplace of both sides [mult both sides by e^{-st} and integrate $t=0$ to $t=\infty$]

• $\mathcal{L}(x'')+3\mathcal{L}(x')+2\mathcal{L}(x) = \mathcal{L}(\frac{1}{2})+\mathcal{L}(e^{-3t})$ — Linearity $\mathcal{L}(af+bg) = a\mathcal{L}(f)+b\mathcal{L}(g)$ [integral of a sum = sum of integrals; constants go through the integral si…

• $1\cdot[s^2\mathcal{L}(x)-s\,x(0)-x'(0)]$
$+3[s\mathcal{L}(x)-x(0)]$
$+2[\mathcal{L}(x)] = \frac{1}{2}\mathcal{L}(1)+\mathcal{L}(e^{-3t})$ — Derivative theorem $\mathcal{L}(y') = s\mathcal{L}(y)-y(0)$

• $\underbrace{[s^2+3s+2]}_{\text{Char Equation appears!}}\mathcal{L}(x) = s\,x(0)+x'(0)+3x(0)$
$+\frac{1}{2}\mathcal{L}(1)+\mathcal{L}(e^{-3t})$ — Move terms to right, collect factor $\mathcal{L}(x)$ on left

• $[s^2+3s+2]\mathcal{L}(x) = \dfrac{1}{2s}+\dfrac{1}{s+3}$ — Use $x(0)=0$, $x'(0)=0$, $\mathcal{L}(1)=\frac{1}{s}$, $\mathcal{L}(e^{at})=\frac{1}{s-a}$

• $\mathcal{L}(x) = \dfrac{3s+3}{(2s)(s+3)(s^2+3s+2)}$ — Divide to isolate $\mathcal{L}(x)$ on the left.

$= \dfrac{3/2}{s(s+3)(s+2)}$ — Factor, cancel $s+1$ top and bottom.

End of step 1: Found $\mathcal{L}(x)$ explicitly.

Step 2.

- The objective is to leave $\mathcal{L}(x)$ on the left unchanged, but change the rhs to look like $\mathcal{L}$(something)

Lerch's Theorem will be applied.

$\mathcal{L}(x_1)=\mathcal{L}(x_2) \Rightarrow x_1 = x_2$

i.e., the $\mathcal{L}$ cancels on both sides.

- $$\mathcal{L}(x) = \frac{3/2}{s(s+3)(s+2)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

Theory of partial fractions from college algebra and Calculus I.

$$= \frac{1/4}{s} + \frac{+1/2}{s+3} + \frac{-3/4}{s+2}$$

By Heaviside's method

$$= \frac{1}{4}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(e^{-3t}) - \frac{3}{4}\mathcal{L}(e^{-2t})$$

By $\mathcal{L}(e^{at}) = \frac{1}{s-a}$!

$$= \mathcal{L}\left(\frac{1}{4} + \frac{1}{2}e^{-3t} - \frac{3}{4}e^{-2t}\right)$$

By Linearity (again).

Step 3.

- Apply Lerch's Theorem to find x

$$x(t) = \frac{1}{4} + \frac{1}{2}e^{-3t} - \frac{3}{4}e^{-2t}$$

The $\mathcal{L}$ cancels on each side of the previous relation [equivalent to the inverse Laplace transform]

Transient and steady-state

$x_{ss}(t) = \frac{1}{4}$

$x_{tr}(t) = \frac{1}{2}e^{-3t} - \frac{3}{4}e^{-2t}$

The end!

Def:

$x = x_{ss} + x_{tr}$

and $x_{tr} \to 0$ as $t \to \infty$.

10.1-3 Find $\mathcal{L}(f)$ for $f(t) = e^{2t+1}$

Example. Find $\mathcal{L}(f)$ for $f(t) = e^{-2t+\pi}$

$$\mathcal{L}(f) = \mathcal{L}(e^{-2t}e^{\pi})$$

use $e^{a+b} = e^a e^b$

$$= e^{\pi}\mathcal{L}(e^{-2t})$$

Linearity of $\mathcal{L}$

$$= e^{\pi}\frac{1}{s-(-2)}$$

Tables

$$= \frac{e^{\pi}}{s+2}$$

10.1-5 Find $\mathcal{L}(f)$ for $f(t) = \sinh(t)$.

Hint: $\sinh(u) = \frac{1}{2}e^{u} - \frac{1}{2}e^{-u}$ by definition.

10.1-17 Find $\mathcal{L}(f)$ for $f(t) = \cos^2(2t)$

Hint: The table contains cosines and sines but not $\cos^2(2t)$. By trig identities,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1.$$

Hence,

$$\cos(4t) = 2\cos^2(2t) - 1$$

This identity implies $f(t) = \frac{1}{2} + \frac{1}{2}\cos(4t)$, a sum of terms already present in the table.

10.1-27 Find $f(t)$ given $\mathcal{L}(f) = \frac{3}{s-4}$

Example. Find $f(t)$ given $\mathcal{L}(f) = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s-16}$

$$\begin{aligned}
\mathcal{L}(f) &= \frac{1}{s} + \frac{-2}{s^2} + \frac{4}{s-16} \\
&= \mathcal{L}(1) + (-2)\,\mathcal{L}(t) + 4\,\mathcal{L}(e^{16t}) && \text{by tables} \\
&= \mathcal{L}(1 - 2t + 4e^{16t}) && \text{Linearity} \\
f(t) &= 1 - 2t + 4e^{16t} && \text{Lerch's Thm applied}
\end{aligned}$$

10.2-7 Solve by Laplace method $\begin{cases} x'' + x = \cos(3t) \\ x(0) = 1, \quad x'(0) = 0 \end{cases}$

Details: Apply $\mathcal{L}$ across the DE and use Laplace rules to obtain the equation (see Ex 2 in 10.1)

Fill in the details!

$$\begin{aligned}
\mathcal{L}(x) &= \frac{1}{s^2+1}\left[s + \mathcal{L}(\cos 3t)\right] \\
&= \frac{1}{s^2+1}\left[s + \frac{s}{s^2+9}\right] \\
&= \frac{s}{s^2+1} + \frac{s}{(s^2+1)(s^2+9)} \\
&= \mathcal{L}(\cos t) + \frac{s}{(s^2+1)(s^2+9)}
\end{aligned}$$

To finish, expand the fraction on the right as partial fractions $\frac{as+b}{s^2+1} + \frac{cs+3d}{s^2+9}$ which equals $\mathcal{L}(a\cos t + b\sin t + c\sin 3t + d\cos 3t)$.

10.2-11 Solve by the Laplace method

$$\begin{cases} x' = 2x + y \\ y' = 6x + 3y \\ x(0) = 1, \quad y(0) = -2 \end{cases}$$

solution details: Transform each DE to obtain equations

$$\begin{cases} s\,\mathcal{L}(x) - 1 = 2\,\mathcal{L}(x) + \mathcal{L}(y) \\ s\,\mathcal{L}(y) + 2 = 6\,\mathcal{L}(x) + 3\,\mathcal{L}(y) \end{cases}$$

write as a linear system $A\mathcal{X} = b$ where $\mathcal{X} = \begin{bmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{bmatrix}$ and then solve it to obtain

$$\mathcal{L}(x) = \frac{s-5}{s(s-5)} = \frac{1}{s}$$

$$\mathcal{L}(y) = \frac{-2}{s}$$

Apply Table methods to get $x = 1$, $y = -2$.

10.2-15 project Solve by the Laplace method

$$\begin{cases} x'' + x' + y' + 2x - y = 0 \\ y'' + x' + y' + 4x - 2y = 0 \\ x(0) = y(0) = 1, \quad x'(0) = y'(0) = 0 \end{cases}$$

Hint: Transform the DEs to obtain the system

$$\begin{bmatrix} s^2+s+2 & s-1 \\ s+4 & s^2+s-2 \end{bmatrix} \begin{bmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{bmatrix} = \begin{bmatrix} s+1+1 \\ s+1+1 \end{bmatrix}$$

Solve by Cramer's Rule or equivalent to get $\mathcal{L}(x) = \frac{s^2+3s+}{s(s^2+3s}$ $\mathcal{L}(y)$ = similar. Expand in partial fractions to get the book's answers, e.g., $x = \frac{2}{3} + \frac{1}{3}e^{-3t/2}(\cos\frac{\sqrt{3}t}{2} + \sqrt{3}\sin$

10.3-3 Find $\mathcal{L}(f)$ for $f(t) = e^{-2t}\sin(3\pi t)$

Example. Find $\mathcal{L}(f)$ for $f(t) = e^{-\pi t}\cos(\pi t)$

$$\mathcal{L}(f) = \mathcal{L}(e^{-\pi t}\cos \pi t)$$
$$= \mathcal{L}(\cos \pi t)\Big|_{s \mapsto s+\pi} \quad \text{Shift Theorem}$$
$$= \frac{s}{s^2+\pi^2}\Big|_{s \mapsto s+\pi} \quad \text{Table}$$
$$= \frac{s+\pi}{(s+\pi)^2+\pi^2}$$

10.3-7 Find $f(t)$ given $\mathcal{L}(f) = \dfrac{1}{s^2+4s+4}$

Example. Find $f(t)$ given $\mathcal{L}(f) = \dfrac{1}{s^2+5s+4}$

$$\mathcal{L}(f) = \frac{1}{s^2+5s+4}$$
$$= \frac{1}{(s+1)(s+4)}$$
$$= \frac{A}{s+1} + \frac{B}{s+4} \quad \text{partial fractions}$$
$$= \mathcal{L}(Ae^{-t} + Be^{-4t}) \quad \text{Tables}$$
$$f(t) = Ae^{-t} + Be^{-4t}$$
$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

$A = \frac{1}{3}$, $B = \frac{-1}{3}$ by Heaviside's method.

10.3-19 Find $f(t)$ given $\mathcal{L}(f) = \dfrac{s^2-2s}{s^4+5s^2+4}$.

Hint:
$$\mathcal{L}(f) = \frac{s^2-2s}{(s^2+4)(s^2+1)}$$
$$= \frac{as+2b}{s^2+4} + \frac{cs+d}{s^2+1} \quad \text{partial fraction theory}$$
$$= \mathcal{L}(a\cos 2t + b\sin 2t + c\cos t + d\sin t)$$

The problem thus reduces to computing constants a, b, c, d

10.3-25 Solve by the Laplace method

$$x'' - 4x = 8t, \quad x(0) = 0, \; x'(0) = 0$$

Hint. Transform the DE to get

$$\mathcal{L}(x) = \frac{8}{s^2(s^2-4)}$$
$$= \frac{8}{s^2(s-2)(s+2)}$$
$$= \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s-2} + \frac{d}{s+2} \quad \text{partial fractions}$$
$$= \mathcal{L}(at + b + ce^{2t} + de^{-2t})$$

It remains to show details about the formula for $\mathcal{L}(x)$ and college algebra to find the constants a, b, c, d. By Lerch's Theorem,

$$x = at + b + ce^{2t} + de^{-2t}$$

The book answer uses: $\sinh(2t) = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}$.

Convolution Theorem

Given two functions $f(t)$, $g(t)$ of exponential order, then

$$\mathcal{L}(f(t))\,\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)\,g(t-x)\,dx\right)$$

Example. Solve for $y(t)$ in the equation $\mathcal{L}(y(t)) = \dfrac{1}{s^2(s-1)}$

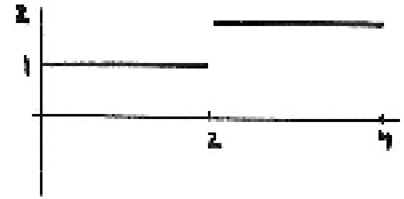
Solution by convolution:

$$\begin{aligned}
\mathcal{L}(y(t)) &= \frac{1}{s^2}\,\frac{1}{s-1} \\
&= \mathcal{L}(t)\,\mathcal{L}(e^t) \\
&= \mathcal{L}\left(\int_0^t x\,e^{t-x}\,dx\right) \\
&= \mathcal{L}\left(e^t\int_0^t x e^{-x}\,dx\right) \\
&= \mathcal{L}\left(e^t(1-e^{-t}-te^{-t})\right) \\
&= \mathcal{L}(e^t-1-t)
\end{aligned}$$

$\therefore$ $\boxed{y(t) = e^t - 1 - t}$ by Lerch's cancellation law.

Check:
$$\begin{aligned}
\mathcal{L}(y) &= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \\
&= \frac{-s+1-s^2+s+s^2}{s^2(s-1)} \\
&= \frac{1}{s^2(s-1)}
\end{aligned}$$

Example. Calculate $\mathcal{L}(f)$ for the periodic extension of $H(t)+H(t-2)$ on $0 \le t \le 4$, of period 4.



The base function is a step, $f=1$ on $0 \le t < 2$, $f=2$ on $2 \le t \le 4$.

$P=4$
$f(t) = 1 + H(t-2)$

$$\begin{aligned}
\mathcal{L}(f) &= \frac{\int_0^P e^{-st} f(t)\,dt}{1-e^{-Ps}} \\
&= \frac{\int_0^4 e^{-st}(1+H(t-2))\,dt}{1-e^{-Ps}} \\
&= \left(\int_0^4 e^{-st}\,dt + \int_2^4 e^{-st}(2)\,dt\right) \Big/ (1-e^{-4s}) \\
&= \frac{1+e^{-2s}-2e^{-4s}}{s(1-e^{-4s})}
\end{aligned}$$

Final answer

Remark

The integration above used

$$\begin{aligned}
\int_0^4 e^{-st} f(t)\,dt &= \int_0^2 e^{-st} f(t)\,dt + \int_2^4 e^{-st} f(t)\,dt \\
&= \int_0^2 e^{-st}(1)\,dt + \int_2^4 e^{-st}(2)\,dt
\end{aligned}$$

Generally, step function integrations require each a splitup of integrals, and subsequent replacement of simple expressions within the integrand, in order to be successfully integrated.

Historical origins of The Laplace Transform

1821 Jean-Baptiste Joseph Fourier publishes "The analytical Theory of heat" in Paris.

Studied heat conduction, for an insulated bar packed on the ends with ice, the heat $u(x,t)$ at position x $(0 \le x \le \pi)$ and time $t \ge 0$ is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t}$$

- Fourier claimed that any initial heat distribution $u(x,0) = f(x)$ could be written as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$ Fourier series

- Dirichlet made the work rigorous by providing hypotheses on f that made it true (1804–1859).

- Fourier's ideas were applied to vibrations of strings, with the vibration excursion from equilibrium $u(x,t)$ again being represented in Fourier's natural sine-cosine coordinate system.

- The Fourier Integral was invented to handle continuous spectra and non-periodic behavior.

$$f(x) = \int_0^{\infty} \left(A(w)\cos(wx) + B(w)\sin(wx)\right) dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)\,dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)\,dv$$

- The Complex Fourier Integral was derived from Euler's classic formula $e^{i\theta} = \cos\theta + i\sin\theta$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{iwx}\,dw$$

$$F(w) = \int_{-\infty}^{\infty} f(x) e^{-iwx}\,dx$$ Fourier Transform

1890

- Oliver Heaviside invented an operational method for solving differential equations, very mysterious. The explanation of why it worked leads to the following mathematical object:

Heaviside's operational calculus evolved into modern Laplace Theory.

$$\int_0^{\infty} e^{-st} f(t)\,dt$$ The Laplace transform

For a function $f(t) = 0$ for $t < 0$, this is the same as

$$\int_{-\infty}^{\infty} e^{-iwt} f(t)\,dt$$ The Fourier transform revisited

with $\underset{\sim}{iw}$ replaced by $\underset{\sim}{s}$.

- Lerch proved a cancellation law that allowed the special transform to be used as an alternate model for a differential equation. It reads:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \Rightarrow f(t) = g(t).$$