Systems of Differential Equations

Elementary Methods

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Translating a Scalar System to a Vector-Matrix System

Consider the scalar system

$$\begin{aligned} u_1'(t) &= 2u_1(t) + 3u_2(t), \\ u_2'(t) &= 4u_1(t) + 5u_2(t). \end{aligned}$$

Define

$$\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}.$$

Then matrix multiply rules imply that the scalar system is equivalent to the vector-matrix equation

$$\mathbf{u}' = A\mathbf{u}$$

Solving a Triangular System

An illustration. Let us solve $\mathbf{u}' = A\mathbf{u}$ for a triangular matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The matrix equation $\mathbf{u}' = A\mathbf{u}$ represents two differential equations:

$$\begin{aligned} u_1' &= u_1, \\ u_2' &= 2u_1 + u_2. \end{aligned}$$

The first equation $u_1' = u_1$ has solution $u_1 = c_1 e^t$. The second equation becomes

$$u_2' = 2c_1 e^t + u_2,$$

which is a first order linear differential equation with solution $u_2 = (2c_1 t + c_2)e^t$. The general solution of $\mathbf{u}' = A\mathbf{u}$ is

$$u_1 = c_1 e^t, \quad u_2 = 2c_1 t e^{-t} + c_2 e^t.$$

Solving a System $\mathbf{u}' = A\mathbf{u}$ with Non-Triangular A

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be non-triangular. Then both $b \neq 0$ and $c \neq 0$ must be satisfied. The scalar form of the system $\mathbf{u}' = A\mathbf{u}$ is

$$\begin{aligned} u_1' &= au_1 + bu_2, \\ u_2' &= cu_1 + du_2. \end{aligned}$$

Theorem 1 (Solving Non-Triangular $\mathbf{u}' = A\mathbf{u}$)
Solutions u_1, u_2 of $\mathbf{u}' = A\mathbf{u}$ are linear combinations of the list of atoms obtained from the roots r of the quadratic equation

$$\det(A - rI) = 0.$$

Proof of the Non-Triangular Theorem

The method is to differentiate the first equation, then use the equations to eliminate u_2, u_2'. This results in a second order differential equation for u_1. The same differential equation is satisfied also for u_2. The details:

$u_1'' = au_1' + bu_2'$	Differentiate the first equation.
$= au_1' + bcu_1 + bdu_2$	Use equation $u_2' = cu_1 + du_2$.
$= au_1' + bcu_1 + d(u_1' - au_1)$	Use equation $u_1' = au_1 + bu_2$.
$= (a + d)u_1' + (bc - ad)u_1$	Second order equation for u_1 found

The characteristic equation is $r^2 - (a + d)r + (bc - ad) = 0$, which is exactly the expansion of $\det(A - rI) = 0$. The proof is complete.

How to Solve a Non-Triangular System $\mathbf{u}' = A\mathbf{u}$

- **Finding u_1.** The two roots r_1, r_2 of the characteristic equation produce two solution atoms,

  In case the roots are distinct, the solution atoms are $e^{r_1 t}$, $e^{r_2 t}$. Then u_1 is a linear combination of atoms: $u_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

- **Finding u_2.** Isolate u_2 in the first differential equation by division:

$$u_2 = \frac{1}{b}(u_1' - a u_1).$$

The two formulas for u_1, u_2 represent the general solution of the system $\mathbf{u}' = A\mathbf{u}$, when A is 2×2.

A Non-Triangular Illustration

Let us solve $\mathbf{u}' = A\mathbf{u}$ when A is the non-triangular matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial is $\det(A - rI) = (1-r)^2 - 4 = (r+1)(r-3)$. Euler's theorem implies solution atoms e^{-t}, e^{3t}. Then u_1 is a linear combination of the solution atoms, $u_1 = c_1 e^{-t} + c_2 e^{3t}$.
The first equation $u_1' = u_1 + 2u_2$ implies

$$\begin{aligned} u_2 &= \frac{1}{2}(u_1' - u_1) \\ &= -c_1 e^{-t} + c_2 e^{3t}. \end{aligned}$$

The general solution of $\mathbf{u}' = A\mathbf{u}$ is then

$$u_1 = c_1 e^{-t} + c_2 e^{3t}, \quad u_2 = -c_1 e^{-t} + c_2 e^{3t}.$$