Electrical Circuits

- Voltage drop formulas of Faraday, Ohm, Coulomb.
- Kirchhoff's laws.
- LRC Circuit equation.
- Electrical-Mechanical Analogy.
- Transient and Steady-state Currents.
- Reactance and Impedance.
- Time lag.
- Electrical Resonance.

Voltage Drop Formulas

Faraday's Law $V_L = L\frac{dI}{dt}$

$L =$ inductance in henries,

$I =$ current in amperes.

Ohm's Law $V_R = RI$

$R =$ resistance in ohms.

Coulomb's Law $V_C = \frac{Q}{C}$

$Q =$ charge in coulombs,

$C =$ capacitance in farads.

Kirchhoff's Laws

The **charge** Q and **current** I are related by the equation

$$\frac{dQ}{dt} = I.$$

- **Loop Law**: *The algebraic sum of the voltage drops around a closed loop is zero.*
- **Junction Law**: *The algebraic sum of the currents at a node is zero.*

LRC Circuit Equation in Charge form

The first law of Kirchhoff implies the RLC circuit equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

where inductor L, resistor R and capacitor C are in a single loop having electromotive force $E(t)$.

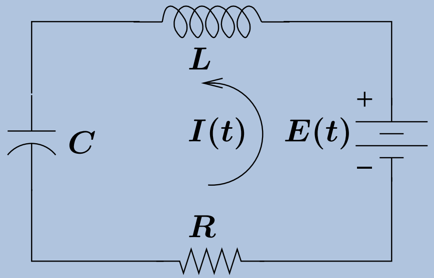

Figure 1. An LRC Circuit.

The components are a resistor R, inductor L, capacitor C and emf $E(t)$. Current $I(t)$ is assigned counterclockwise direction, from minus to plus on the emf terminals.

LRC Circuit Equation in Current Form

Differentiation of the charge form of the LRC circuit equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

gives the current form of the LRC circuit equation

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}.$$

Electrical–Mechanical Analogy

$$
\begin{array}{rcrcrcl}
mx'' & + & cx' & + & kx & = & F(t), \\
LQ'' & + & RQ & + & C^{-1}Q & = & E(t).
\end{array}
$$

Table 1. Electrical–Mechanical Analogy

Mechanical System	Electrical System
Mass m	Inductance L
Dampening constant c	Resistance R
Hooke's constant k	Reciprocal capacitance $1/C$
Position x	Charge Q [or Current I]
External force F	Electromotive force E [or dE/dt]

Transient and Steady-state Currents

The theory of mechanical systems leads to electrical results by applying the electrical-mechanical analogy to the LRC circuit equation in current form with $E(t) = E_0 \sin \omega t$. We assume L, R and C positive.

- The solution I_h of the homogeneous equation $LI'' + RI' + \frac{1}{C}I = 0$ is a **transient current**, satisfying
$$\lim_{t\to\infty} I_h(t) = 0.$$
- The non-homogeneous equation $LI'' + RI' + \frac{1}{C}I = E_0\omega \cos \omega t$ has a unique periodic solution [**steady-state current**]
$$I_{\text{ss}}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}}, \quad S \equiv \omega L - \frac{1}{\omega C}, \quad \tan \alpha = \frac{\omega RC}{1 - LC\omega^2}.$$
It is found by the method of undetermined coefficients.

Reactance and Impedance

Write

$$I_{\mathrm{SS}}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}}$$

as

$$I_{\mathrm{SS}}(t) = \frac{E_0}{Z} \cos(\omega t - \alpha)$$

where

$$Z = \sqrt{R^2 + S^2} \quad \text{is called the } \textbf{impedance}$$

$$S = \omega L - \frac{1}{\omega C} \quad \text{is called the } \textbf{reactance.}$$

Time Lag

The steady-state current $I_{\text{SS}}(t)\frac{E_0}{Z}\cos(\omega t - \alpha)$ can be written as a sine function using trigonometric identity $\cos(x - \pi/2) = \sin(x)$ with $\alpha = \delta + \pi/2$:

$$I_{\text{SS}}(t) = \frac{E_0}{Z}\sin(\omega t - \delta), \quad \tan\delta = \frac{LC\omega^2 - 1}{\omega RC} = \frac{S}{R}.$$

Because the input is

$$E(t) = E_0\omega\sin(\omega t),$$

then the **time lag** between the input voltage and the steady-state current is

$$\frac{\delta}{\omega} = \frac{\arctan(S/R)}{\omega} \quad \text{seconds.}$$

Electrical Resonance

Resonance in an LRC circuit is defined only for sinusoidal inputs $E(t) = E_0 \sin(\omega t)$. Then the differential equation in current form is

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{E_0 \omega}{L} \cos(\omega t).$$

Resonance happens if there is a frequency ω which maximizes the steady-state solution amplitude $I_0 = E_0/Z$, $Z = \sqrt{R^2 + S^2}$, $S = \omega L - \dfrac{1}{C\omega}$. By calculus, this happens exactly when $dZ/d\omega = 0$, which gives the **resonant frequency**

$$\omega = \frac{1}{\sqrt{LC}}.$$

Details: $dI_0/d\omega = 0$ if and only if $-E_0 Z^{-2} \frac{dZ}{d\omega} = 0$, which is equivalent to $\frac{dZ}{d\omega} = 0$. Then $2S \frac{dS}{d\omega} = 0$ and finally $S = 0$, because $\frac{dS}{d\omega} > 0$. The equation $S = 0$ is equivalent to $\omega = 1/\sqrt{LC}$.