An RREF Method for Finding Inverses

An efficient method to find the inverse $\boldsymbol{B}$ of a square matrix $\boldsymbol{A}$, should it happen to exist, is to form the augmented matrix $\boldsymbol{C} = < \boldsymbol{A}|\boldsymbol{I} >$ and then read off $\boldsymbol{B}$ as the package of the last $\boldsymbol{n}$ columns of $\mathbf{rref}(\boldsymbol{C}) = < \boldsymbol{I}|\boldsymbol{B} >$. This method is based upon the equivalence

$$\mathbf{rref}(< \boldsymbol{A}|\boldsymbol{I} >) = < \boldsymbol{I}|\boldsymbol{B} > \quad \text{if and only if} \quad \boldsymbol{AB} = \boldsymbol{I}.$$

Main Results

Theorem 1 (Inverse Test)
If A and B are square matrices such that $AB = I$, then also $BA = I$. Therefore, only one of the equalities $AB = I$ or $BA = I$ is required to check an inverse.

Theorem 2 (The rref Inversion Method)
Let A and B denote square matrices. Then

(a) If $\mathrm{rref}(< A|I >) =< I|B >$, then $AB = BA = I$ and B is the inverse of A.

(b) If $AB = BA = I$, then $\mathrm{rref}(< A|I >) =< I|B >$.

(c) If $\mathrm{rref}(< A|I >) =< C|B >$ and $C \neq I$, then A is not invertible.

Finding inverses

The $\mathbf{rref}$ inversion method will be illustrated for the matrix

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Define the first frame of the sequence to be $C_1 = < C|I >$, then compute the frame sequence to $\mathbf{rref}(C)$ as follows.

$$C_1 = \left(\begin{array}{ccr|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

First Frame

$$C_2 = \left(\begin{array}{ccr|crc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right)$$

`combo(3,2,-1)`

$$C_3 = \left(\begin{array}{ccr|crr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

`mult(3,1/2)`

$$C_4 = \left(\begin{array}{ccc|crr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

`combo(3,2,1)`

`combo(3,1,-1)`

$$C_5 = \left(\begin{array}{ccc|crr} 1 & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

Last Frame

Extract the Inverse Matrix

The theory

$$\mathbf{rref}(<\boldsymbol{A}|\boldsymbol{I}>) =<\boldsymbol{I}|\boldsymbol{B}> \quad \text{if and only if} \quad \boldsymbol{AB}=\boldsymbol{I}$$

implies that the inverse of $\boldsymbol{A}$ is the matrix in the right panel of the last frame

$$\boldsymbol{C}_5 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right).$$

Then

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix}.$$