Applications of Systems of Differential Equations

- Home Heating
- Newton Cooling Model
- Homogeneous Solution and Particular Solution
- Underpowered Heater
- Forced Air Furnace and Furnace Cycling

Home Heating

Consider a typical home with attic, basement and insulated main floor.

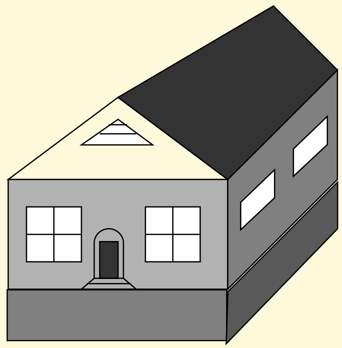

Figure 1. Typical home with attic and basement. The below-grade basement and the attic are un-insulated. Only the main living area is insulated.

Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

We will analyze the changing temperatures in the three levels using Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

Newton Cooling Model

Assume it is winter time and the outside temperature in constantly 35°F during the day. Also assumed is a basement earth temperature of 45°F. Initially, the heat is off for several days. The initial values at noon ($t = 0$) are then $x(0) = 45$, $y(0) = z(0) = 35$.

> **Portable heater**. A small electric heater is turned on at noon, with thermostat set for 100°F. When the heater is running, it provides a 20°F rise per hour, therefore it takes some time to reach 100°F (probably never!).

Newton's cooling law

Temperature rate = k(Temperature difference)

will be applied to five boundary surfaces: (0) the basement walls and floor, (1) the basement ceiling, (2) the main floor walls, (3) the main floor ceiling, and (4) the attic walls and ceiling. Newton's cooling law gives positive cooling constants k_0, k_1, k_2, k_3, k_4 and the equations

$$
\begin{aligned}
x' &= k_0(45 - x) + k_1(y - x), \\
y' &= k_1(x - y) + k_2(35 - y) + k_3(z - y) + 20, \\
z' &= k_3(y - z) + k_4(35 - z).
\end{aligned}
$$

Insulation Constants and the Final Model

The insulation constants will be defined as

$$k_0 = 1/2, \quad k_1 = 1/2, \quad k_2 = 1/4, \quad k_3 = 1/4, \quad k_4 = 3/4$$

to reflect insulation quality. The reciprocal $1/k$ is approximately the amount of time in hours required for 63% of the temperature difference to be exchanged. For instance, 4 hours elapse for the main floor. The model:

$$\begin{aligned}
x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\
y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\
z' &= \frac{1}{4}(y - z) + \frac{3}{4}(35 - z).
\end{aligned}$$

Homogeneous Solution

The homogeneous solution in vector form is given in terms of constants

$$\boldsymbol{a = 1 + \sqrt{5})/4, \;\; b = 1 - \sqrt{5})/4}$$

and arbitrary constants $\boldsymbol{c_1}$, $\boldsymbol{c_2}$, $\boldsymbol{c_3}$ by the formula

$$\begin{pmatrix} x_h(t) \\ y_h(t) \\ z_h(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix}.$$

Particular Solution

A particular solution is an equilibrium solution

$$\begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix}.$$

The homogeneous solution has limit zero at infinity, hence the temperatures of the three spaces hover around $x = 56.4$, $y = 67.7$, $z = 43.2$ degrees Fahrenheit. Specific information can be gathered by solving for c_1, c_2, c_3 according to the initial data $x(0) = 45$, $y(0) = z(0) = 35$. The answers are

$$c_1 = 5, \quad c_2 = -\frac{25}{2} + \frac{7}{2}\sqrt{5}, \quad c_3 = -\frac{25}{2} - \frac{7}{2}\sqrt{5}.$$

Underpowered Heater

To the main floor each hour is added 20°F, but the heat escapes at a substantial rate, so that after one hour $y \approx 68$°F. After five hours, $y \approx 68$°F. The heater in this example is so inadequate that even after many hours, the main living area is still under 69°F.

Forced air furnace

Replacing the space heater by a normal furnace adds the difficulty of **switches** in the input, namely, the thermostat turns off the furnace when the main floor temperature reaches $70°$F, and it turns it on again after a $4°$F temperature drop. We will suppose that the furnace has four times the BTU rating of the space heater, which translates to an $80°$F temperature rise per hour.

The study of the forced air furnace requires two differential equations, one with 20 replaced by 80 (DE 1, furnace on) and the other with 20 replaced by 0 (DE 2, furnace off). The plan is to use the first differential equation on time interval $0 \le t \le t_1$, then switch to the second differential equation for time interval $t_1 \le t \le t_2$. The time intervals are selected so that $y(t_1) = 70$ (the thermostat setting) and $y(t_2) = 66$ (thermostat setting less 4 degrees). Numerical work gives the following results.

Furnace Cycling

Time in minutes	Main floor temperature	Model	Furnace
31.6	**70**	DE 1	on
40.9	**66**	DE 2	off
45.3	**70**	DE 1	on
54.6	**66**	DE 2	off

The reason for the non-uniform times between furnace cycles can be seen from the model. Each time the furnace cycles, heat enters the main floor, then escapes through the other two levels. Consequently, the initial conditions applied to models 1 and 2 are changing, resulting in different solutions to the models on each switch.