Constant Coefficient Equations

Theorem 1 (First Order Recipe)
Let a and b be constants, $a \neq 0$. Let r_1 denote the root of $ar + b = 0$. Then $y = c_1 e^{r_1 x}$ is the general solution of the first order equation

$$ay' + by = 0.$$

Theorem 2 (Second Order Recipe)
Let $a \neq 0$, b and c be real constants. Let r_1, r_2 be the two roots of $ar^2 + br + c = 0$, real or complex. If complex, then let $r_1 = \overline{r_2} = \alpha + i\beta$ with $\beta > 0$. Consider the following three cases, organized by the sign of the discriminant $D = b^2 - 4ac$:

$D > 0$ (Real distinct roots)	$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}.$
$D = 0$ (Real equal roots)	$y_1 = e^{r_1 x}, \quad y_2 = x e^{r_1 x}.$
$D < 0$ (Conjugate roots)	$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x).$

Then y_1, y_2 are two solutions of

$$ay'' + by' + cy = 0$$

and all solutions are given by $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitrary constants.

Theorem 3 (Existence-Uniqueness)
Let the coefficients $a(x)$, $b(x)$, $c(x)$, $f(x)$ be continuous on an interval J containing $x = x_0$. Assume $a(x) \neq 0$ on J. Let y_0 and y_1 be constants. The initial value problem

$$\begin{aligned} &a(x)y'' + b(x)y' + c(x)y = f(x), \\ &y(x_0) = y_0, \quad y'(x_0) = y_1 \end{aligned}$$

has a unique solution $y(x)$ defined on J.

Theorem 4 (Superposition)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has the *superposition property*:

> If y_1, y_2 are solutions and c_1, c_2 are constants, then the combination $y(x) = c_1y_1(x) + c_2y_2(x)$ is a solution.

Theorem 5 (Homogeneous Structure)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1, c_2 are arbitrary constants and $y_1(x)$, $y_2(x)$ are solutions.

Theorem 6 (Non-Homogeneous Structure)
The non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$ has general solution

$$y(x) = y_h(x) + y_p(x),$$

where

$y_h(x)$ is the general solution of the homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$, and

$y_p(x)$ is a particular solution of the nonhomogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$.