Linear Equation Method

$$y' + p(x)y = q(x)$$

- Write the DE in standard form $y' + p(x)y = q(x)$. The method applies only if this is possible!
- Evaluate and simplify $P(x) = \int p(x)dx$. Choose the constant of integration to simplify e^{P}.
- Replace $y' + p(x)y$ by $\dfrac{(e^{P}y)'}{e^{P}}$
- Clear fractions. Apply the method of quadrature.

Variation of parameters Formula

$$y' + p(x)y = q(x)$$

$$y = y_h + y_p \quad \text{Superposition}$$

Homogeneous solution.

$$y_h = c\,e^{-P(x)} \qquad c = \text{constant}$$

$$P(x) = \int p(x)dx$$

Particular solution.

$$y_p = e^{-P(x)} \int q(r)e^{P(r)}dr$$

2.2 Separable Equations

An equation $y' = f(x, y)$ is called **separable** provided algebraic operations, usually multiplication, division and factorization, allow it to be written in the form $y' = F(x)G(y)$ for some functions F and G. This class includes the *quadrature equations* $y' = F(x)$. Separable equations and associated solution methods were discovered by G. Leibniz in 1691 and formalized by J. Bernoulli in 1694.

It is important to emphasize that only certain operations are allowed to convert an equation $y' = f(x, y)$ into **separable form** $y' = F(x)G(y)$. Generally speaking, addition and subtraction of terms is disallowed, although it may help in factorization. The first operations to try are *division* and *multiplication*.

Non-separability tests.

Test I — Equation $y' = f(x, y)$ is **non-separable** if for some pair of points (x_0, y_0), (x, y) in the domain of f

(1) $$f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0.$$

Test II — The equation $y' = f(x, y)$ is non-separable if either $f_x(x, y)/f(x, y)$ depends on y or $f_y(x, y)/f(x, y)$ depends on x.

To illustrate, equation $y' = xy + y^2$ is non-separable by *Test I*, because $f(x, 1)f(0, y) - f(0, 1)f(x, y) = (x + 1)y^2 - (xy + y^2) = x(y^2 - y) \neq 0$. To apply *Test II*, observe that $f_x/f = 1/(x + y)$ is not constant as a function of y.

To justify *Test I*, assume $f(x, y) = F(x)G(y)$, then equation (1) fails because each term on the left side of (1) evaluates to $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y).

To justify *Test II*, assume $f(x, y) = F(x)G(y)$ and F, G are sufficiently differentiable. Then $f_x(x, y)/f(x, y) = F'(x)/F(x)$ is independent of y and $f_y(x, y)/f(x, y) = G'(y)/G(y)$ is independent of x.

Finding the separable form. An equation $y' = f(x, y)$ is **separable** provided the following identity holds for some (x_0, y_0) in the domain of f:

(2) $$f(x, y) = \frac{f(x, y_0)}{f(x_0, y_0)} f(x_0, y), \quad \text{for all } (x, y).$$

If $y' = f(x, y)$ is separable, then a **separable form** $y' = F(x)G(y)$ can be found from the formulas

(3) $$F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \quad G(y) = f(x_0, y).$$

Separation Test. A **separated equation** $y'/G(y) = F(x)$ is recognized by this test:

Left Side. The left side of the equation has factor y' and it is independent of x.

Right side. The right side of the equation is independent of y and y'.

Variables-Separable Method. Determined by the method are the following kinds of solution formulas.

Equilibrium Solutions. They are the constant solutions of the differential equation. For a separable equation $y' = F(x)G(y)$, they are found by solving $G(y) = 0$.

Non-Equilibrium Solutions. For separable equation $y' = F(x)G(y)$, it is a solution y with $G(y) \neq 0$.

The term *equilibrium* is borrowed from kinematics. Alternative terms are **rest solution** and **stationary solution**; all mean $y' = 0$ in calculus terms.

It is important to *check the solution* to a separable equation, because certain steps used to arrive at the "solution" may not be reversible.

For most applications, the two kinds of solutions suffice to determine all possible solutions. In general, a separable equation may have non-unique solutions to some initial value problem. To prevent this from happening, it can be assumed that F, G and G' are continuous; see the Picard-Lindelöf theorem, page 87. If non-uniqueness does occur, then often the equilibrium and non-equilibrium solutions can be pieced together to represent all solutions.

Finding Equilibria. The search for equilibria can be done without finding the separable form of $y' = f(x, y)$. It is enough to solve for y in the equation $f(x, y) = 0$. An equilibrium solution y cannot depend upon x, because it is *constant*. If y turns out to depend on x, after solving $f(x, y) = 0$ for y, then this is sufficient evidence that $y' = f(x, y)$ is **not separable**. Some examples:

$y' = y\sin(x - y)$	It is *not separable*. The solutions of $y\sin(x-y) = 0$ are $y = 0$ and $x-y = n\pi$ for any integer n. The solution $y = x - n\pi$ is non-constant, therefore the equation cannot be separable.
$y' = xy(1 - y^2)$	It is *separable*. The equation $xy(1 - y^2) = 0$ has