2.4 Undetermined Coefficients

Studied here is the subject of undetermined coefficients for linear first order differential equations. The method applies to $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters (reviewed in an example and in exercises). The requirements and limitations:

> **1**. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.
>
> **2**. The function $r(x)$ is a sum of constants times Euler solution atoms.

Definition 7 (Euler Solution Atom)
An **Euler base atom** is a term having one of the forms

$$1,\ e^{ax},\ \cos bx,\ \sin bx,\ e^{ax}\cos bx \quad \text{or} \quad e^{ax}\sin bx.$$

The symbols a and b are real constants, with $a \neq 0$ and $b > 0$.

An **Euler solution atom** equals x^n(Euler base atom). Symbol $n \geq 0$ is an integer.

Examples. The terms x^3, $x\cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, if $r(x) = 4\sin x + 5xe^x$, then split the sum into terms and drop the coefficients 4 and 5 to identify atoms $\sin x$ and xe^x; then $r(x)$ is a sum of constants times atoms.

The Method.

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
2. **Correction rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching atoms e^{-px}, xe^{-px}, ... by xe^{-px}, x^2e^{-px}, ... (other atoms appearing in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match coefficients of atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Illustrated. We will solve

$$y' + 2y = xe^x + 2x + 1 + 3\sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (split terms and drop coefficients to identify new atoms). Because the solution e^{-2x} of $y' + 2y = 0$ does not appear in the list of atoms, then the correction rule does not apply. The corrected trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x.$$

Then

$$\begin{aligned} r(x) &= y' + 2y \\ &= d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x \\ &\quad + 2d_1 + 2d_2x + 2d_3e^x + 2d_4xe^x + 2d_5\cos x + 2d_6\sin x \\ &= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\ &\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x) \end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term 1 in $r(x)$ matches the constant term in the expansion of $y' + 2y$, giving $1 = d_2 + 2d_1$. Writing out the matches, and swapping sides, gives the equations

$$\begin{array}{rrrrrrl} 2d_1 + & d_2 & & & & & = 1, \\ & 2d_2 & & & & & = 2, \\ & & 3d_3 + & d_4 & & & = 0, \\ & & & 3d_4 & & & = 1, \\ & & & & 2d_5 + & d_6 & = 0, \\ & & & & - \quad d_5 + & 2d_6 & = 3. \end{array}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule (reviewed in Chapter 3) to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

Remarks. The method of matching coefficients of atoms left and right is a subject of linear algebra, called *linear independence*. The method works because any finite list of atoms is known to be linearly independent. Further details for this technical topic appear in this text's linear algebra chapters.

A Correction Rule Illustration. Solve the equation

$$y' + 3y = 8e^x + 3x^2e^{-3x}$$

by the method of undetermined coefficients. Verify that the general solution $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = 2e^x + x^3e^{-3x}.$$

Solution: The right side $r(x) = 8e^x + 3x^2e^{-3x}$ is constructed from atoms e^x, x^2e^{-3x}. Repeated differentiation of these atoms identifies the new list of atoms e^x, e^{-3x}, xe^{-3x}, x^2e^{-3x}. The correction rule applies because the solution e^{-3x} of $y'+3y = 0$ appears in the list. The atoms of the form x^me^{-3x} are multiplied by x to give the new list of atoms e^x, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}. Readers should take note that atom e^x is unaffected by the correction rule modification. Then the corrected trial solution is

$$y = d_1e^x + d_2xe^{-3x} + d_3x^2e^{-3x} + d_4x^3e^{-3x}.$$

The trial solution expression y is substituted into $y' + 3y = 2e^x + x^2e^{-3x}$ to give the equation

$$4d_1e^x + d_2e^{-3x} + 2d_3xe^{-3x} + 3d_4x^2e^{-3x} = 8e^x + 3x^2e^{-3x}.$$

Coefficients of atoms on each side of the preceding equation are matched to give the equations

$$\begin{aligned} 4d_1 \qquad\qquad\qquad &= 8, \\ d_2 \qquad\qquad &= 0, \\ 2d_3 \qquad &= 0, \\ 3d_4 &= 3. \end{aligned}$$

Then $d_1 = 2$, $d_2 = d_3 = 0$, $d_4 = 1$ and the particular solution is reported to be $y_p = 2e^x + x^3e^{-3x}$.

Remarks on the Method of Undetermined Coefficients

A mystery for the novice is the construction of the trial solution. **Why should it work**? Explained here is the reason behind the method of repeated differentiation to find the atoms in the trial solution.

The theory missing is that the general solution y of $y' + py = r(x)$ is a sum of constants times atoms (under the cited **limitations**). We don't try to prove this result, but use it to motivate the method.

The theory reduces the question of finding a trial solution to finding a sum of constants times Euler solution atoms. The question is: *which atoms*?

Consider this example: $y' - 3y = e^{3x} + xe^x$. The answer for y is revealed by finding a sum of constants times atoms such that y' and $-3y$ add termwise to $e^{3x} + xe^x$. The requirement eliminates all atoms from consideration except those containing exponentials e^{3x} and e^x.

Initially, we have to consider infinitely many atoms $e^{3x}, xe^{3x}, x^2e^{3x}, \ldots$ and $e^x, xe^x, x^2e^x, \ldots$. Such terms would also appear in y', but adding terms of this type to get $r(x) = e^{3x} + xe^x$ requires only the smaller list $e^{3x}, xe^{3x}, e^x, xe^x$. We have cut down the number of terms in y to four or less!

The algorithm presented here together with the correction rule strips down the number of terms to a minimum. Further details of the method appear in the section on higher order equations.

Examples

19 Example (Variation of Parameters Method) Solve the equation $2y' + 6y = 4xe^{-3x}$ by the method of variation of parameters, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = x^2e^{-3x}.$$

Solution: Divide the equation by 2 to obtain the standard linear form

$$y' + 3y = 2xe^{-3x}.$$

Solution y_h. The homogeneous equation $y' + 3y = 0$ is solved by the shortcut formula $y_h = \dfrac{\text{constant}}{\text{integrating factor}}$ to give $y_h = ce^{-3x}$.

Solution y_p. Identify $p(x) = 3$, $r(x) = 2xe^{-3x}$ from the standard form. The mechanics: let $y' = f(x, y) \equiv 2xe^{-3x} - 3y$ and define $r(x) = f(x, 0)$, $p(x) = -f_y(x, y) = 3$. The variation of parameters formula is applied as follows. First, compute the integrating factor $W(x) = e^{\int p(x)dx} = e^{3x}$. Then

$$\begin{aligned} y_p(x) &= (1/W(x)) \int r(x)W(x)dx \\ &= e^{-3x} \textstyle\int 2xe^{-3x}e^{3x}dx \\ &= x^2e^{-3x}. \end{aligned}$$

It must be explained that all integration constants were set to zero, in order to obtain the shortest possible expression for y_p. Indeed, if $W = e^{3x+c_1}$ instead of e^{3x}, then the factors $1/W$ and W contribute constant factors $1/e^{c_1}$ and e^{c_1}, which multiply to one; the effect is to set $c_1 = 0$. On the other hand, an integration constant c_2 added to $\int r(x)W(x)dx$ adds the homogeneous solution c_2e^{-3x} to the expression for y_p. Because we seek the shortest expression which is a solution to the non-homogeneous differential equation, the constant c_2 is set to zero.

20 Example (Undetermined Coefficient Method) Solve the equation $2y' + 6y = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ by the method of undetermined coefficients, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Solution: The method applies, because the differential equation $2y' + 6y = 0$ has constant coefficients and the right side $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ is constructed from the list of atoms xe^{-x}, xe^{-3x}, $\sin x$.

List of Atoms. Differentiate the atoms in $r(x)$, namely xe^{-x}, xe^{-3x}, $\sin x$, to find the new list of atoms e^{-x}, xe^{-x}, e^{-3x}, xe^{-3x}, $\cos x$, $\sin x$. The solution e^{-3x} of $2y' + 6y = 0$ appears in the list: the correction rule applies. Then e^{-3x}, xe^{-3x} are replaced by xe^{-3x}, x^2e^{-3x} to give the corrected list of atoms e^{-x}, xe^{-x}, xe^{-3x}, x^2e^{-3x}, $\cos x$, $\sin x$. Please note that only two of the six atoms were corrected.

Trial solution. The corrected trial solution is

$$y = d_1e^{-x} + d_2xe^{-x} + d_3xe^{-3x} + d_4x^2e^{-3x} + d_5\cos x + d_6\sin x.$$

Substitute y into $2y' + 6y = r(x)$ to give

$$\begin{aligned} r(x) &= 2y' + 6y \\ &= (4d_1 + 2d_2)e^{-x} + 4d_2xe^{-x} + 2d_3e^{-3x} + 4d_4xe^{-3x} \\ &\quad +(2d_6 + 6d_5)\cos x + (6d_6 - 2d_5)\sin x. \end{aligned}$$

Equations. Matching atoms on the left and right of $2y' + 6y = r(x)$, given $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$, justifies the following equations for the undetermined coefficients; the solution is $d_2 = 1$, $d_1 = -1/2$, $d_3 = 0$, $d_4 = 1$, $d_6 = 3/4$, $d_5 = -1/4$.

$$\begin{aligned} 4d_1 + 2d_2 \qquad\qquad &= 0, \\ 4d_2 \qquad\qquad &= 4, \\ 2d_3 \qquad &= 0, \\ 4d_4 \quad &= 4, \\ 6d_5 + 2d_6 &= 0, \\ -2d_5 + 6d_6 &= 5. \end{aligned}$$

Report. The trial solution upon substitution of the values for the undetermined coefficients becomes

$$y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Exercises 2.4

Variation of Parameters I. Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rW}{W}, \quad W = e^{\int p(x)dx}.$$

1. $y' = x + 1$

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

6. $y' - y = 1$

7. $2y' + y = e^x$

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

10. $xy' = 1 - x^2$

Variation of Parameters II. Compute the particular solution given by

$$y_p^*(x) = \frac{\int_{x_0}^{x} rW}{W(x)}, \quad W(t) = e^{\int_{x_0}^{t} p(x)dx}.$$

11. $y' = x + 1$, $x_0 = 0$

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^{x}$, $x_0 = 1$

18. $2y' + y = e^{-x}$, $x_0 = 1$

19. $xy' = x + 1$, $x_0 = 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Euler Solution Atoms. Report the list of distinct Euler atoms of the given function $f(x)$. Then $f(x)$ is a sum of constants times the Euler atoms from this list.

21. $x + e^x$

22. $1 + 2x + 5e^x$

23. $x(1 + x + 2e^x)$

24. $x^2(2 + x^2) + x^2e^{-x}$

25. $\sin x \cos x + e^x \sin 2x$

26. $\cos^2 x - \sin^2 x + x^2e^x \cos 2x$

27. $(1 + 2x + 4x^5)e^x e^{-3x} e^{x/2}$

28. $(1+2x+4x^5+e^x \sin 2x)e^{-3x/4}e^{x/2}$

29. $\dfrac{x + e^x}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^x \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution. Differentiate repeatedly $f(x)$ and report the list of distinct Euler solution atoms which appear in f and all its derivatives. Then each derivative of $f(x)$ is a sum of constants times the Euler atoms from this list.

31. $12 + 5x^2 + 6x^7$

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^x$

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^x + \cos 2x$

36. $(x + e^x) \sin x + (x - e^{-x}) \cos 2x$

37. $(x + e^x + \sin 3x + \cos 2x)e^{-2x}$

38. $(x^2e^{-x} + 4 \cos 3x + 5 \sin 2x)e^{-3x}$

39. $(1 + x^2)(\sin x \cos x - \sin 2x)e^{-x}$

40. $(8 - x^3)(\cos^2 x - \sin^2 x)e^{3x}$

Correction Rule. Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the correction rule.

41. $y_h(x) = ce^{2x}$, $y = d_1 + d_2x + d_3e^{2x}$

42. $y_h(x) = ce^{2x}$, $y = d_1 + d_2e^{2x} + d_3xe^{2x}$

43. $y_h(x) = ce^{0x}$, $y = d_1 + d_2x + d_3x^2$

44. $y_h(x) = ce^{x}$, $y = d_1 + d_2x + d_3x^2$

45. $y_h(x) = ce^{x}$, $y = d_1 \cos x + d_2 \sin x + d_3e^x$

46. $y_h(x) = ce^{2x}$, $y = d_1e^{2x} \cos x + d_2e^{2x} \sin x$

47. $y_h(x) = ce^{2x}$, $y = d_1e^{2x} + d_2xe^{2x} + d_3x^2e^{2x}$

48. $y_h(x) = ce^{-2x}$, $y = d_1e^{-2x} + d_2xe^{-2x} + d_3e^{2x} + d_4xe^{2x}$

49. $y_h(x) = cx^2$, $y = d_1 + d_2x + d_3x^2$

50. $y_h(x) = cx^3$, $y = d_1 + d_2x + d_3x^2$

Undetermined Coefficients: Trial Solution. Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y' = x^3 + 5 + x^2e^x(3 + 2x + \sin 2x)$

52. $y' = x^2 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

53. $y' - y = x^3 + 2x + 5 + x^4e^x(2 + 4x + 7\cos 2x)$

54. $y' - y = x^4 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

55. $y' - 2y = x^3 + x^2 + x^3e^x(2e^x + 3x + 5\sin 4x)$

56. $y' - 2y = x^3e^{2x} + x^2e^x(3 + 4e^x + 2\cos 2x)$

57. $y' + y = x^2 + 5x + 2 + x^3e^{-x}(6x + 3\sin x + 2\cos x)$

58. $y' - 2y = x^5 + 5x^3 + 14 + x^3e^x(5 + 7xe^{-3x})$

59. $2y' + 4y = x^4 + 5x^5 + 2x^8 + x^3e^x(7 + 5xe^x + 5\sin 11x)$

60. $5y' + y = x^2 + 5x + 2e^{x/5} + x^3e^{x/5}(7 + 9x + 2\sin(9x/2))$

Undetermined Coefficients. Compute a particular solution y_p according to the method of undetermined coefficients. Report (1) the initial trial solution, (2) the corrected trial solution, (3) the system of equations for the undetermined coefficients and finally (4) the formula for y_p.

61. $y' + y = x + 1$

62. $y' + y = 2x - 1$

63. $y' - y = e^x + e^{-x}$

64. $y' - y = xe^x + e^{-x}$

65. $y' - 2y = 1 + x + e^{2x} + \sin x$

66. $y' - 2y = 1 + x + xe^{2x} + \cos x$

67. $y' + 2y = xe^{-2x} + x^3$

68. $y' + 2y = (2 + x)e^{-2x} + xe^x$

69. $y' = x^2 + 4 + xe^x(3 + \cos x)$

70. $y' = x^2 + 5 + xe^x(2 + \sin x)$