The Unique Solution Case

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Frame Sequence Defined

Imagine watching an expert, who applies swap, multiply and combination rules to a system of equations, in order to find the solution. At each application of *swap*, *combo* or *mult*, the system of equations is re-written onto a new piece of paper.

The content of each completed paper is photographed, to produce a **frame** in a sequence of camera snapshots. The first frame is the original system and the last frame gives the solution to the system of equations.

Eliminated from the sequence are all arithmetic details, which are expected to be supplied by the reader. This is not a video of the solving process, but a sequence of selected video frames which documents major steps.

Table 1. A Frame Sequence.

Frame 1	Frame 2	Frame 3
Original System $\begin{cases} x - y = 2, \\ 3y = -3. \end{cases}$	Apply `mult(2,1/3)` $\begin{cases} x - y = 2, \\ y = -1. \end{cases}$	Apply `combo(2,1,1)` $\begin{cases} x = 1, \\ y = -1. \end{cases}$

Lead Variables

In an equation, a variable is called **leading** provided it is the first variable, in variable list order, appearing with non-zero coefficient, called the **leading coefficient**.
In a system of equations, a **lead variable** is a variable such that:

1. The variable appears exactly once in the entire system.

2. In the unique equation where it appears, the variable is leading with coefficient one.

Free Variables

A **free variable** is a non-lead variable. A variable which does not appear in the system is a free variable. Detection of a free variable otherwise must be from a system in which every non-zero equation has a lead variable.

Example

The $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ system below has free variables $\boldsymbol{y}$, $\boldsymbol{z}$ and lead variable $\boldsymbol{x}$.

$$\begin{aligned} \boldsymbol{x} + \mathbf{2}\boldsymbol{z} &= \mathbf{2}, \\ \mathbf{0} &= \mathbf{0}, \\ \mathbf{0} &= \mathbf{0}. \end{aligned}$$

Unique Solution

A consistent system in which **every variable is a lead variable** must have a **unique solution**. The system must look like the final frame of this sequence:

Table 2. Unique solution case.

Frame 1	Frame 2	Frame 3
Original System $\begin{cases} x - y = 2, \\ 3y = -3. \end{cases}$	Apply `mult(2,1/3)` $\begin{cases} x - y = 2, \\ y = -1. \end{cases}$	Apply `combo(2,1,1)` $\begin{cases} x = 1, \\ y = -1. \end{cases}$

In the last frame, *all* of the variables appear, in variable list order. To the left of the equal sign each variable appears just once, with coefficient one, and to the right of the equal sign appear numbers.

Solving for a Unique Solution

To solve a system with a unique solution, we apply the toolkit operations of swap, multiply and combination (acronyms `swap, mult, combo`), one operation per frame, until the last frame displays the unique solution.

Because all variables will be lead variables in the last frame, we seek to create a new lead variable in each frame. Sometimes, this is not possible, even if it is the general objective. Exceptions are swap and multiply operations, which are often used to prepare for creation of a lead variable. Listed in Table 3 are the rules and conventions that we use to create frame sequences.

Table 3. Conventions and rules for creating frame sequences.

Order of Variables. Variables in equations appear in variable list order to the left of the equal sign.

Order of Equations. Equations are listed in variable list order inherited from their lead variables. Equations without lead variables appear next. Equations without variables appear last. Multiple swap operations convert any system to this convention.

New Lead Variable. Select a new lead variable as the *first variable*, in variable list order, which appears among the equations without a lead variable.

An Illustration

$$\begin{array}{rcrcrcr} & & y & + & 4z & = & 2, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}$$

1 Original system.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 4z & = & 2. \end{array}$$

2 `swap(1,3)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 4z & = & 2. \end{array}$$

3 `combo(1,2,-1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & z & = & 1. \end{array}$$

4 `combo(2,3,1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & & y & + & 3z & = & 1, \\ & & & & z & = & 1. \end{array}$$

5 `mult(2,-1)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & z & = & 1. \end{array}$$

6 `combo(2,1,-2)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & & & = & -2, \\ & & & & z & = & 1. \end{array}$$

7 `combo(3,2,-3)`

$$\begin{array}{rcrcrcr} x & & & & & = & 5, \\ & & y & & & = & -2, \\ & & & & z & = & 1. \end{array}$$

8 `combo(3,1,3)`
Last Frame.
Unique solution.