Chapter 9

Eigenanalysis

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This presentation of **matrix eigenanalysis** treats the subject in depth for a 3×3 matrix A. The generalization to an $n \times n$ matrix A is easily supplied by the reader.

9.1 Eigenanalysis I

Treated here is eigenanalysis for matrix equations. The topics are *eigenanalysis*, *eigenvalue*, *eigenvector*, *eigenpair* and *diagonalization*.

What's Eigenanalysis?

Matrix eigenanalysis is a computational theory for the matrix equation $\mathbf{y} = A\mathbf{x}$. Here, we assume A is a 3×3 matrix.

The basis of eigenanalysis is **Fourier's Model**:

$$
\begin{array}{rcl}
\mathbf{x} & = & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \text{ implies} \\
\mathbf{y} & = & A\mathbf{x} \\
 & = & c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.
\end{array} \tag{1}
$$

These relations can be written as a single equation:

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

The scale factors λ_1, λ_2, λ_3 and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ depend only on A. Symbols c_1, c_2, c_3 stand for arbitrary numbers. This implies variable $\mathbf{x}$ exhausts all possible 3-vectors in $\mathcal{R}^3$ and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is a basis for $\mathcal{R}^3$. Fourier's model is a replacement process:

> To compute $A\mathbf{x}$ from $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, replace each vector $\mathbf{v}_i$ by its scaled version $\lambda_i\mathbf{v}_i$.

Fourier's model is said to **hold** provided there exist λ_1, λ_2, λ_3 and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ satisfying (1). It is known that Fourier's model fails for certain matrices A, for example,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Powers and Fourier's Model. Equation (1) applies to compute powers A^n of a matrix A using only the basic vector space toolkit. To illustrate, only the vector toolkit for $\mathcal{R}^3$ is used in computing

$$A^5\mathbf{x} = x_1\lambda_1^5\mathbf{v}_1 + x_2\lambda_2^5\mathbf{v}_2 + x_3\lambda_3^5\mathbf{v}_3.$$

This calculation does not depend upon finding previous powers A^2, A^3, A^4 as would be the case by using matrix multiply.

Fourier's model can reduce computational effort. Matrix 3×3 multiplication to produce $\mathbf{y}_k = A^k\mathbf{x}$ requires $9k$ multiply operations whereas Fourier's 3×3 model gives the answer with $3k + 9$ multiply operations.

Fourier's model illustrated. Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = -5,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\begin{array}{rcccccccc} \mathbf{x} & = & c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & + & c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} & + & c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} & \text{implies} \\ A\mathbf{x} & = & c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & + & c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} & + & c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} & \end{array}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\mathbf{y} = A\mathbf{x}$ by scaling independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, which is a triad or **coordinate system**.

Success stories for eigenanalysis include geometric problems, systems of differential equations representing mechanical systems, chemical kinetics, electrical networks, and heat and wave partial differential equations.

In summary:

The subject of **eigenanalysis** discovers a coordinate system and scale factors such that Fourier's model holds. Fourier's model simplifies the matrix equation $\mathbf{y} = A\mathbf{x}$.

Differential Equations and Fourier's Model. Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{array}{rcrcrcl} x_1' &=& x_1 &+& 3x_2, & & \\ x_2' &=& & & 2x_2 &-& x_3, \\ x_3' &=& & & & - & 5x_3. \end{array}$$

The matrix form is $\mathbf{x}' = A\mathbf{x}$, where A is the same matrix used in the Fourier model illustration of the previous paragraph.

Fourier's idea of re-scaling applies as well to differential equations, in the following context. First, expand the initial condition $\mathbf{x}(0)$ in terms of basis elements $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$:

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3.\mathbf{v}_3.$$

Then the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by replacing each $\mathbf{v}_i$ by the re-scaled vector $e^{\lambda_i t}\mathbf{v}_i$, giving the formula

$$\mathbf{x}(t) = c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + c_3e^{\lambda_3 t}\mathbf{v}_3.$$

For the illustration here, the result is

$$\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) = c_1e^t \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right) + c_2e^{2t} \left(\begin{array}{c} 3 \\ 1 \\ 0 \end{array}\right) + c_3e^{-5t} \left(\begin{array}{c} 1 \\ -2 \\ -14 \end{array}\right).$$

What's an Eigenvalue?

It is a scale factor. An eigenvalue is also called a *proper value* or a *hidden value*. Symbols λ_1, λ_2, λ_3 used in Fourier's model are eigenvalues.

Historically, the German term *eigenwert* was used exclusively in literature, because German was the preferred publication language for physics. Due to literature migration into English language journals, a hybrid term *eigenvalue* evolved, the German word *wert* replaced by *value*

A Key Example. Let $\mathbf{x}$ in $\mathcal{R}^3$ be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit $\mathbf{x}$-data into proper MKS units (meters-kilogram-second) $\mathbf{y}$-data via the equations

$$\begin{array}{rcl} y_1 &=& x_1, \\ y_2 &=& 0.001x_2, \\ y_3 &=& 0.01x_3. \end{array} \tag{2}$$

Equations (2) are a **model** for changing units. Scaling factors $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$ are the **eigenvalues** of the model. To summarize:

> The **eigenvalues** of a model are **scale factors**. They are normally represented by symbols λ_1, λ_2, λ_3, ...

The data conversion problem (2) can be represented as $\mathbf{y} = A\mathbf{x}$, where the diagonal matrix A is given by

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = \frac{1}{1000},\ \lambda_3 = \frac{1}{100}.$$

What's an Eigenvector?

Symbols $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors*. They are assumed independent.

The **eigenvectors** $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of model (2) are three independent **directions of application** for the respective scale factors $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$. The directions identify the components of the data set, to which the individual scale factors are to be multiplied, to perform the data conversion. Because the scale factors apply individually to the x_1, x_2 and x_3 components of a vector $\mathbf{x}$, then

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

The data is represented as $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$. The answer $\mathbf{y} = A\mathbf{x}$ is given by the equation

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} \lambda_1 x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda_3 x_3 \end{pmatrix} \\ &= \lambda_1 x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= x_1\lambda_1\mathbf{v}_1 + x_2\lambda_2\mathbf{v}_2 + x_3\lambda_3\mathbf{v}_3. \end{aligned}$$

In summary:

> The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

History of Fourier's Model. The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\mathbf{y} = A\mathbf{x}$.

> The vector $\mathbf{y} = A\mathbf{x}$ is obtained from eigenvalues λ_1, λ_2, $\ldots$, λ_n and eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\mathbf{v}_1$, $\lambda_2\mathbf{v}_2$, $\ldots$, $\lambda_n\mathbf{v}_n$:
>
> $$\begin{array}{rcccccccc} \mathbf{x} & = & c_1\mathbf{v}_1 & + & c_2\mathbf{v}_2 & + & \cdots & + & c_n\mathbf{v}_n \quad \text{implies} \\ \mathbf{y} & = & x_1\lambda_1\mathbf{v}_1 & + & x_2\lambda_2\mathbf{v}_2 & + & \cdots & + & c_n\lambda_n\mathbf{v}_n. \end{array}$$

Determining Equations. The eigenvalues and eigenvectors are determined by homogeneous matrix–vector equations. In Fourier's model

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$$

choose $c_1 = 1$, $c_2 = c_3 = 0$. The equation reduces to $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Similarly, taking $c_1 = c_2 = 0$, $c_2 = 1$ implies $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. Finally, taking $c_1 = c_2 = 0$, $c_3 = 1$ implies $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$. This proves:

Theorem 1 (Determining Equations in Fourier's Model)
Assume Fourier's model holds. Then the eigenvalues and eigenvectors are determined by the three equations

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_1\mathbf{v}_1, \\ A\mathbf{v}_2 &= \lambda_2\mathbf{v}_2, \\ A\mathbf{v}_3 &= \lambda_3\mathbf{v}_3. \end{aligned}$$

The three relations of the theorem can be distilled into one homogeneous matrix–vector equation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Write it as $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$, then replace $\lambda\mathbf{x}$ by $\lambda I\mathbf{x}$ to obtain the standard form[1]

$$(A - \lambda I)\mathbf{v} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0}.$$

Let $B = A - \lambda I$. The equation $B\mathbf{v} = \mathbf{0}$ has a nonzero solution $\mathbf{v}$ if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occurs if and only if $\mathbf{rref}(B)$ has a row of zeros. Determinant theory gives a more concise statement: $\det(B) = 0$ if and only if $B\mathbf{v} = \mathbf{0}$ has infinitely many solutions. This proves:

Theorem 2 (Characteristic Equation)
If Fourier's model holds, then the eigenvalues λ_1, λ_2, λ_3 are roots λ of the polynomial equation

$$\det(A - \lambda I) = 0.$$

[1] Identity I is required to factor out the matrix $A - \lambda I$. It is wrong to factor out $A - \lambda$, because A is 3×3 and λ is 1×1, incompatible sizes for matrix addition.

The equation is called the **characteristic equation**. The **characteristic polynomial** is the polynomial on the left, normally obtained by cofactor expansion or the triangular rule.

An Illustration.

$$\begin{aligned}\det\left(\begin{pmatrix}1&3\\1&2\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right) &= \begin{vmatrix}1-\lambda & 3\\ 1 & 2-\lambda\end{vmatrix}\\ &= (1-\lambda)(2-\lambda)-6\\ &= \lambda^2-3\lambda-4\\ &= (\lambda+1)(\lambda-4).\end{aligned}$$

The characteristic equation $\lambda^2-3\lambda-4=0$ has roots $\lambda_1=-1$, $\lambda_2=4$. The characteristic polynomial is $\lambda^2-3\lambda-4$.

Theorem 3 (Finding Eigenvectors of A)
For each root λ of the characteristic equation, write the frame sequence for $B=A-\lambda I$ with last frame $\mathbf{rref}(B)$, followed by solving for the general solution $\mathbf{v}$ of the homogeneous equation $B\mathbf{v}=\mathbf{0}$. Solution $\mathbf{v}$ uses invented parameter names t_1, t_2, ... The vector basis answers $\partial_{t_1}\mathbf{v}$, $\partial_{t_2}\mathbf{v}$, ... are independent **eigenvectors** of A paired to eigenvalue λ.

Proof: The equation $A\mathbf{v}=\lambda\mathbf{v}$ is equivalent to $B\mathbf{v}=\mathbf{0}$. Because $\det(B)=0$, then this system has infinitely many solutions, which implies the frame sequence starting at B ends with $\mathbf{rref}(B)$ having at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols t_1, t_2, ..., and then the vector basis is obtained by from the corresponding list of partial derivatives. Each basis element is a nonzero solution of $A\mathbf{v}=\lambda\mathbf{v}$. By construction, the basis elements (eigenvectors for λ) are collectively independent. The proof is complete.

The theorem implies that a 3×3 matrix A with eigenvalues 1, 2, 3 causes three frame sequences to be computed, each sequence producing one eigenvector. In contrast, if A has eigenvalues 1, 1, 1, then only one frame sequence is computed.

Definition 1 (Eigenpair)
An **eigenpair** is an eigenvalue λ together with a matching eigenvector $\mathbf{v}\neq\mathbf{0}$ satisfying the equation $A\mathbf{v}=\lambda\mathbf{v}$. The pairing implies that scale factor λ is applied to direction $\mathbf{v}$.

A 3×3 matrix A for which Fourier's model holds has eigenvalues λ_1, λ_2, λ_3 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. The **eigenpairs** of A are

$$(\lambda_1,\mathbf{v}_1),(\lambda_2,\mathbf{v}_2),(\lambda_3,\mathbf{v}_3).$$

Theorem 4 (Independence of Eigenvectors)
If $(\lambda_1,\mathbf{v}_1)$ and $(\lambda_2,\mathbf{v}_2)$ are two eigenpairs of A and $\lambda_1\neq\lambda_2$, then $\mathbf{v}_1$, $\mathbf{v}_2$ are independent.

More generally, if $(\lambda_1, \mathbf{v}_1)$, ..., $(\lambda_k, \mathbf{v}_k)$ are eigenpairs of A corresponding to distinct eigenvalues $\lambda_!$, ..., λ_k, then $\mathbf{v}_1$, ..., $\mathbf{v}_k$ are independent.

Proof: Let's solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ for c_1, c_2. Apply A to this equation, then $c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0}$. Multiply the first equation by λ_2 and subtract from the second equation to get $c_1(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$. Because $\lambda_1 \neq \lambda_2$, cancellation gives $c_1\mathbf{v}_1 = \mathbf{0}$. The assumption $\mathbf{v}_1 \neq \mathbf{0}$ implies $c_1 = 0$. Similarly, $c_2 = 0$. This proves $\mathbf{v}_1$, $\mathbf{v}_2$ are independent.

The general case is proved by induction on k. The case $k = 1$ follows because a nonzero vector is an independent set. Assume it holds for $k-1$ and let's prove it for k, when $k > 1$. We solve

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

for c_1, ..., c_k. Apply A to this equation, which effectively replaces each c_i by $\lambda_i c_i$. Then multiply the first equation by λ_1 and subtract the two equations to get

$$c_2(\lambda_1 - \lambda_2)\mathbf{v}_1 + \cdots + c_k(\lambda_1 - \lambda_k)\mathbf{v}_k = \mathbf{0}.$$

By the induction hypothesis, all coefficients are zero. Because $\lambda_1 - \lambda_i \neq 0$ for $i > 1$, then c_2 through c_k are zero. Return to the first equation to obtain $c_1\mathbf{v}_1 = \mathbf{0}$. Because $\mathbf{v}_1 \neq \mathbf{0}$, then $c_1 = 0$. This finishes the induction.

Definition 2 (Diagonalizable Matrix)
A square matrix A for which Fourier's model holds is called **diagonalizable**. The $n \times n$ matrix A has n eigenpairs with independent eigenvectors.

Eigenanalysis Facts.

1. An **eigenvalue** λ of a triangular matrix A is one of the diagonal entries. If A is non-triangular, then an eigenvalue is found as a root λ of $\det(A - \lambda I) = 0$.

2. An eigenvalue of A can be zero, positive, negative or even complex. It is a pure number, with a physical meaning inherited from the model, e.g., a scale factor.

3. An **eigenvector** for eigenvalue λ (a scale factor) is a nonzero direction $\mathbf{v}$ of application satisfying $A\mathbf{v} = \lambda\mathbf{v}$. It is found from a frame sequence starting at $B = A - \lambda I$ and ending at $\mathbf{rref}(B)$. Independent eigenvectors are computed from the general solution as partial derivatives $\partial/\partial t_1$, $\partial/\partial t_2$, ...

4. If a 3×3 matrix has three independent eigenvectors, then they collectively form in $\mathcal{R}^3$ a **basis** or **coordinate system**.

Eigenpair Packages

The eigenpairs of a 3×3 matrix for which Fourier's model holds are labeled

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2), \quad (\lambda_3, \mathbf{v}_3).$$

An **eigenvector package** is a matrix package $\mathcal{P}$ of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ given by

$$\mathcal{P} = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).$$

An **eigenvalue package** is a matrix package $\mathcal{D}$ of eigenvalues given by

$$\mathcal{D} = \mathbf{diag}(\lambda_1, \lambda_2, \lambda_3).$$

Important is the *pairing* that is inherited from the eigenpairs, which dictates the packaging order of the eigenvectors and eigenvalues. Matrices $\mathcal{P}$, $\mathcal{D}$ are **not unique**: possible are 3! (= 6) column permutations.

An Example. The eigenvalues for the data conversion problem (2) are $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$ and the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the columns of the identity matrix I, given by (3). Then the eigenpair packages are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 5 (Eigenpair Packages)
Let $\mathcal{P}$ be a matrix package of eigenvectors and $\mathcal{D}$ the corresponding matrix package of eigenvalues. Then for all vectors $\mathbf{c}$,

$$A\mathcal{P}\mathbf{c} = \mathcal{P}\mathcal{D}\mathbf{c}.$$

Proof: The result is valid for $n \times n$ matrices. We prove it for 3×3 matrices. The two sides of the equation are expanded as follows.

$$\begin{aligned}
\mathcal{P}\mathcal{D}\mathbf{c} &= \mathcal{P} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} && \text{Expand RHS.} \\
&= \mathcal{P} \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \lambda_3 c_3 \end{pmatrix} \\
&= \lambda_1 c_1 \mathbf{v}_1 + \lambda_2 c_2 \mathbf{v}_2 + \lambda_3 c_3 \mathbf{v}_3 && \text{Because } \mathcal{P} \text{ has columns } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3. \\
A\mathcal{P}\mathbf{c} &= A(c_1 \mathbf{v}_2 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3) && \text{Expand LHS.} \\
&= c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + c_3 \lambda_3 \mathbf{v}_3 && \text{Fourier's model.}
\end{aligned}$$

The Equation $AP = PD$

The question of Fourier's model holding for a given 3×3 matrix A is settled here. According to the result, a matrix A for which Fourier's model holds can be constructed by the formula $A = PDP^{-1}$ where D is any diagonal matrix and P is an invertible matrix.

Theorem 6 ($AP = PD$)
Fourier's model $A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$ holds for eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $(\lambda_3, \mathbf{v}_3)$ if and only if the packages

$$P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \quad D = \mathbf{diag}(\lambda_1, \lambda_2, \lambda_3)$$

satisfy the two requirements

1. Matrix P is invertible, e.g., $\det(\mathcal{P}) \neq 0$.
2. Matrix $A = PDP^{-1}$, or equivalently, $AP = PD$.

Proof: Assume Fourier's model holds. Define $P = \mathcal{P}$ and $D = \mathcal{D}$, the eigenpair packages. Then **1** holds, because the columns of P are independent. By Theorem 5, $AP\mathbf{c} = PD\mathbf{c}$ for all vectors $\mathbf{c}$. Taking $\mathbf{c}$ equal to a column of the identity matrix I implies the columns of AP and PD are identical, that is, $AP = PD$. A multiplication of $AP = PD$ by P^{-1} gives **2**.

Conversely, let P and D be given packages satisfying **1**, **2**. Define $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ to be the columns of P. Then the columns pass the rank test, because P is invertible, proving independence of the columns. Define λ_1, λ_2, λ_3 to be the diagonal elements of D. Using $AP = PD$, we calculate the two sides of $AP\mathbf{c} = PD\mathbf{c}$ as in the proof of Theorem 5, which shows that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_2\mathbf{v}_3$ implies $A\mathbf{x} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$. Hence Fourier's model holds.

The Matrix Eigenanalysis Method

The preceding discussion of data conversion now gives way to abstract definitions which distill the essential theory of eigenanalysis. All of this is algebra, devoid of motivation or application.

Definition 3 (Eigenpair)
A pair $(\lambda, \mathbf{v})$, where $\mathbf{v} \neq \mathbf{0}$ is a vector and λ is a complex number, is called an **eigenpair** of the $n \times n$ matrix A provided

$$A\mathbf{v} = \lambda\mathbf{v} \quad (\mathbf{v} \neq \mathbf{0} \text{ required}). \tag{4}$$

The **nonzero** requirement in (4) results from seeking directions for a coordinate system: the zero vector is not a direction. Any vector $\mathbf{v} \neq \mathbf{0}$ that satisfies (4) is called an **eigenvector** for λ and the value λ is called an **eigenvalue** of the square matrix A. The algorithm:

Theorem 7 (Algebraic Eigenanalysis)
Eigenpairs $(\lambda, \mathbf{v})$ of an $n \times n$ matrix A are found by this two-step algorithm:

> **Step 1 (College Algebra)**. Solve for eigenvalues λ in the nth order polynomial equation $\det(A - \lambda I) = 0$.
>
> **Step 2 (Linear Algebra)**. For a given root λ from **Step 1**, a corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$ is found by applying the **rref** method[2] to the homogeneous linear equation
>
> $$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$
>
> The reported answer for $\mathbf{v}$ is routinely the list of partial derivatives $\partial_{t_1}\mathbf{v}$, $\partial_{t_2}\mathbf{v}$, ..., where t_1, t_2, ... are invented symbols assigned to the free variables.

The reader is asked to apply the algorithm to the identity matrix I; then **Step 1** gives n duplicate answers $\lambda = 1$ and **Step 2** gives n answers, the columns of the identity matrix I.

Proof: The equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$, which is a set of homogeneous equations, consistent always because of the solution $\mathbf{v} = \mathbf{0}$.

Fix λ and define $B = A - \lambda I$. We show that an eigenpair $(\lambda, \mathbf{v})$ exists with $\mathbf{v} \neq \mathbf{0}$ if and only if $\det(B) = 0$, i.e., $\det(A - \lambda I) = 0$. There is a unique solution $\mathbf{v}$ to the homogeneous equation $B\mathbf{v} = \mathbf{0}$ exactly when Cramer's rule applies, in which case $\mathbf{v} = \mathbf{0}$ is the unique solution. All that Cramer's rule requires is $\det(B) \neq 0$. Therefore, an eigenpair exists exactly when Cramer's rule fails to apply, which is when the determinant of coefficients is zero: $\det(B) = 0$.

Eigenvectors for λ are found from the general solution to the system of equations $B\mathbf{v} = 0$ where $B = A - \lambda I$. The **rref** method produces systematically a reduced echelon system from which the general solution $\mathbf{v}$ is written, depending on invented symbols $t_1, \ldots, t_k$. Since there is never a unique solution, at least one free variable exists. In particular, the last frame $\mathbf{rref}(B)$ of the sequence has a row of zeros, which is a useful sanity test.

The **basis of eigenvectors** for λ is obtained from the general solution $\mathbf{v}$, which is a linear combination involving the parameters $t_1, \ldots, t_k$. The **basis elements** are reported as the list of partial derivatives $\partial_{t_1}\mathbf{v}, \ldots, \partial_{t_k}\mathbf{v}$.

Diagonalization

A square matrix A is called **diagonalizable** provided $AP = PD$ for some diagonal matrix D and invertible matrix P. The preceding discussions imply that D must be a package of eigenvalues of A and P must be the corresponding package of eigenvectors of A. The requirement on P

[2]For $B\mathbf{v} = \mathbf{0}$, the frame sequence begins with B, instead of $\mathbf{aug}(B, \mathbf{0})$. The sequence ends with $\mathbf{rref}(B)$. Then the reduced echelon system is written, followed by assignment of free variables and display of the general solution $\mathbf{v}$.

to be invertible is equivalent to asking that the eigenvectors of A be independent and equal in number to the column dimension of A.

The matrices A for which Fourier's model is valid is precisely the class of diagonalizable matrices. This class is not the set of all square matrices: there are matrices A for which Fourier's model is invalid. They are called **non-diagonalizable matrices**.

Theorem 4 implies that the construction for eigenvector package P always produces independent columns. Even if A has fewer than n eigenpairs, the construction still produces independent eigenvectors. In such **non-diagonalizable** cases, caused by insufficient columns to form P, matrix A must have an eigenvalue of multiplicity greater than one.

If all eigenvalues are distinct, then the correct number of independent eigenvectors were found and A is then **diagonalizable** with packages D, P satisfying $AP = PD$. This proves the following result.

Theorem 8 (Distinct Eigenvalues)
If an $n \times n$ matrix A has n distinct eigenvalues, then it has n eigenpairs and A is diagonalizable with eigenpair packages D, P satisfying $AP = PD$.

Examples

1 Example (Computing 2×2 Eigenpairs)
Find all eigenpairs of the 2×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

Solution:
College Algebra. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= (1-\lambda)(-1-\lambda) && \text{Sarrus' rule.} \end{aligned}$$

Linear Algebra. The eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Details:
Eigenvector for $\lambda_1 = 1$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 0 \\ 2 & -1-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} && \text{Swap and multiply rules.} \\ &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = t_1$, $y = t_1$ is eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = -1$.

$$\begin{aligned} A - \lambda_2 I &= \begin{pmatrix} 1-\lambda_2 & 0 \\ 2 & -1-\lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\ &= \mathbf{rref}(A - \lambda_2 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = 0$, $y = t_1$ is eigenvector $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2 Example (Computing 2×2 Complex Eigenpairs)

Find all eigenpairs of the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Solution:

College Algebra. The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= (1-\lambda)^2 + 4 && \text{Sarrus' rule.} \end{aligned}$$

The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2 + 4 = 0$. Alternatively, use $(1-\lambda)^2 = -4$ and take square roots.

Linear Algebra. The eigenpairs are $\left(1+2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$, $\left(1-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$.

Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 2 \\ -2 & 1-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \\ &\approx \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} && \text{Multiply rule.} \\ &\approx \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix} && \text{Combination rule, multiplier}=-i. \\ &\approx \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} && \text{Swap rule.} \\ &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = -it_1$, $y = t_1$ is eigenvector $\mathbf{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 1 - 2i$.
The problem $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$ has solution $\mathbf{v} = \overline{\mathbf{v}}_1$, because taking conjugates across the equation gives $(A - \overline{\lambda}_2 I)\overline{\mathbf{v}} = \mathbf{0}$; then $\lambda_1 = \overline{\lambda}_2$ gives $\mathbf{v} = \overline{\mathbf{v}}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

3 Example (Computing 3×3 Eigenpairs)
Find all eigenpairs of the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution:
College Algebra. The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= ((1-\lambda)^2 + 4)(3-\lambda) && \text{Cofactor rule and Sarrus' rule.} \end{aligned}$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2+4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

Linear Algebra.
The eigenpairs are $\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.
Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 2 & 0 \\ -2 & 1-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{pmatrix} \\ &\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule.} \\ &\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Combination rule, factor}=-i. \\ &\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\ &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector $\mathbf{v}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 1 - 2i$.

The problem $(A-\lambda_2 I)\mathbf{v}_2 = \mathbf{0}$ has solution $\mathbf{v}_2 = \overline{\mathbf{v}}_1$. To see why, take conjugates across the equation to give $(\overline{A}-\overline{\lambda}_2 I)\overline{\mathbf{v}}_2 = \mathbf{0}$. Then $\overline{A} = A$ (A is real) and $\lambda_1 = \overline{\lambda}_2$ gives $(A - \lambda_1 I)\overline{\mathbf{v}}_2 = 0$. Then $\overline{\mathbf{v}}_2 = \mathbf{v}_1$. Finally, $\mathbf{v}_2 = \overline{\overline{\mathbf{v}}}_2 = \overline{\mathbf{v}}_1 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

Eigenvector for $\lambda_3 = 3$.

$$
\begin{aligned}
A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 \\ -2 & 1-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\
&= \mathbf{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
\end{aligned}
$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

4 Example (Decomposition $A = PDP^{-1}$)
Decompose $A = PDP^{-1}$ where P, D are eigenvector and eigenvalue packages, respectively, for the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Write explicitly Fourier's model in vector-matrix notation.

Solution: By the preceding example, the eigenpairs are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

The packages are therefore

$$D = \mathbf{diag}(1+2i, 1-2i, 3), \quad P = \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Fourier's model. The action of A in the model

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$$

is to replace the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ by scaled vectors $\lambda_1\mathbf{v}_1$, $\lambda_2\mathbf{v}_2$, $\lambda_3\mathbf{v}_3$. In vector form, the model is

$$AP\mathbf{c} = PD\mathbf{c}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then the action of A is to replace eigenvector package P by the re-scaled package PD. Explicitly,

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ implies} \\ A\mathbf{x} &= c_1(1+2i) \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2(1-2i) \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3(3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

5 Example (Diagonalization I)
Report **diagonalizable** or **non-diagonalizable** for the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If A is diagonalizable, then assemble and report eigenvalue and eigenvector packages D, P.

Solution: The matrix A is **non-diagonalizable**, because it fails to have 4 eigenpairs. The details:

Eigenvalues.

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} (3-\lambda)^2 && \text{Cofactor expansion applied twice.} \\ &= \left((1-\lambda)^2 + 4\right)(3-\lambda)^2 && \text{Sarrus' rule.} \end{aligned}$$

The roots are $1 \pm 2i$, 3, 3, listed according to multiplicity.

Eigenpairs. They are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right).$$

Because only three eigenpairs exist, instead of four, then the matrix A is **non-diagonalizable**. Details:

Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned}
A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 2 & 0 & 0 \\ -2 & 1-\lambda_1 & 0 & 0 \\ 0 & 0 & 3-\lambda_1 & 1 \\ 0 & 0 & 0 & 3-\lambda_1 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & 2-2i & 1 \\ 0 & 0 & 0 & 2-2i \end{pmatrix} \\
&\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 2-2i & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule, three times.} \\
&\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
&\approx \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
&= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}$$

The general solution is $x_1 = -it_1$, $x_2 = t_1$, $x_3 = 0$, $x_4 = 0$. Then ∂_{t_1} applied to this solution gives the reported eigenpair.

Eigenvector for $\lambda_2 = 1 - 2i$.
Because λ_2 is the conjugate of λ_1 and A is real, then an eigenpair for λ_2 is found by taking the complex conjugate of the eigenpair reported for λ_1.

Eigenvector for $\lambda_3 = 3$. In theory, there can be one or two eigenpairs to report. It turns out there is only one, because of the following details.

$$\begin{aligned}
A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 & 0 \\ -2 & 1-\lambda_3 & 0 & 0 \\ 0 & 0 & 3-\lambda_3 & 1 \\ 0 & 0 & 0 & 3-\lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Multiply rule, two times.}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Combination and multiply rule.}$$

$$= \mathbf{rref}(A - \lambda_3 I) \qquad \text{Reduced echelon form.}$$

Apply ∂_{t_1} to the general solution $x_1 = 0$, $x_2 = 0$, $x_3 = t_1$, $x_4 = 0$ to give the eigenvector matching the eigenpair reported above.

6 Example (Diagonalization II)
Report **diagonalizable** or **non-diagonalizable** for the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If A is diagonalizable, then assemble and report eigenvalue and eigenvector packages D, P.

Solution: The matrix A is **diagonalizable**, because it has 4 eigenpairs

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Then the eigenpair packages are given by

$$D = \begin{pmatrix} -1+2i & 0 & 0 & 0 \\ 0 & 1-2i & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -i & i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The details parallel the previous example, except for the calculation of eigenvectors for $\lambda_3 = 3$. In this case, the reduced echelon form has two rows of zeros and parameters t_1, t_2 appear in the general solution. The answers given above for eigenvectors correspond to the partial derivatives ∂_{t_1}, ∂_{t_2}.

7 Example (Non-diagonalizable Matrices)
Verify that the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are all non-diagonalizable.

Solution: Let A denote any one of these matrices and let n be its dimension.

First, we will decide on diagonalization, without computing eigenpairs. Assume, in order to reach a contradiction, that eigenpair packages D, P exist with D diagonal and P invertible such that $AP = PD$. Because A is triangular, its eigenvalues appear already on the diagonal of A. Only 0 is an eigenvalue and its multiplicity is n. Then the package D of eigenvalues is the zero matrix and an equation $AP = PD$ reduces to $AP = 0$. Multiply $AP = 0$ by P^{-1} to obtain $A = 0$. But A is not the zero matrix, a contradiction. We conclude that A is not diagonalizable.

Second, we attack the diagonalization question directly, by solving for the eigenvectors corresponding to $\lambda = 0$. The frame sequence has first frame $B = A - \lambda I$, but B equals $\mathbf{rref}(B)$ and no computations are required. The resulting reduced echelon system is just $x_1 = 0$, giving $n - 1$ free variables. Therefore, the eigenvectors of A corresponding to $\lambda = 0$ are the last $n - 1$ columns of the identity matrix I. Because A does not have n independent eigenvectors, then A is not diagonalizable.

Similar examples of non-diagonalizable matrices A can be constructed with A having from 1 up to $n - 1$ independent eigenvectors. The examples with ones on the super-diagonal and zeros elsewhere have exactly one eigenvector.

8 Example (Fourier's 1822 Heat Model)
Fourier's 1822 treatise *Théorie analytique de la chaleur* studied dissipation of heat from a laterally insulated welding rod with ends held at 0°C. Assume the initial heat distribution along the rod at time $t = 0$ is given as a linear combination

$$f = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Symbols $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are in the vector space V of all twice continuously differentiable functions on $0 \le x \le 1$, given explicitly as

$$\mathbf{v}_1 = \sin \pi x, \quad \mathbf{v}_2 = \sin 2\pi x, \quad \mathbf{v}_3 = \sin 3\pi x.$$

Fourier's heat model re-scales[3] each of these vectors to obtain the temperature $u(t, x)$ at position x along the rod and time $t > 0$ as the model equation

$$u(t, x) = c_1 e^{-\pi^2 t}\mathbf{v}_1 + c_2 e^{-4\pi^2 t}\mathbf{v}_2 + c_3 e^{-9\pi^2 t}\mathbf{v}_3.$$

Verify that $u(t, x)$ solves Fourier's partial differential equation heat model

$$\begin{array}{lcll} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial^2 u}{\partial x^2}, & \\ u(0, x) & = & f(x), & 0 \le x \le 1, \\ u(t, 0) & = & 0, & \text{zero Celsius at rod's left end,} \\ u(t, 1) & = & 0, & \text{zero Celsius at rod's right end.} \end{array}$$

[3]The scale factors are not constants nor are they eigenvalues, but rather, they are exponential functions of t, as was the case for matrix differential equations $\mathbf{x}' = A\mathbf{x}$

Solution: First, we prove that the partial differential equation is satisfied by Fourier's solution $u(t,x)$. This is done by expanding the left side (LHS) and right side (RHS) of the differential equation, separately, then comparing the answers for equality.

Trigonometric functions $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are solutions of three different linear ordinary differential equations: $u''+\pi^2 u=0$, $u''+4\pi^2 u=0$, $u''+9\pi^2 u=0$. Because of these differential equations, we can compute directly

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 c_1 e^{-\pi^2 t}\mathbf{v}_1 - 4\pi^2 c_2 e^{-4\pi^2 t}\mathbf{v}_2 - 9\pi^2 c_3 e^{-9\pi^2 t}\mathbf{v}_3.$$

Similarly, computing $\partial_t u(t,x)$ involves just the differentiation of exponential functions, giving

$$\frac{\partial u}{\partial t} = -\pi^2 c_1 e^{-\pi^2 t}\mathbf{v}_1 - 4\pi^2 c_2 e^{-4\pi^2 t}\mathbf{v}_2 - 9\pi^2 c_3 e^{-9\pi^2 t}\mathbf{v}_3.$$

Because the second display is exactly the first, then LHS = RHS, proving that the partial differential equation is satisfied.

The relation $u(0,x)=f(x)$ is proved by observing that each exponential factor becomes $e^0=1$ when $t=0$.

The two relations $u(t,0)=u(t,1)=0$ hold because each of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ vanish at $x=0$ and $x=1$. The verification is complete.

Exercises 9.1

Eigenanalysis. Classify as true or false. If false, then correct the text to make it true.

1. The purpose of eigenanalysis is to find a coordinate system.

2. Diagonal matrices have all their eigenvalues on the last row.

3. Eigenvalues are scale factors.

4. Eigenvalues of a diagonal matrix cannot be zero.

5. Eigenvectors $\mathbf{v}$ of a diagonal matrix can be zero.

6. Eigenpairs $(\lambda,\mathbf{v})$ of a diagonal matrix A satisfy the equation $A\mathbf{v}=\lambda\mathbf{v}$.

Eigenpairs of a Diagonal Matrix. Find the eigenpairs of A.

7. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

9. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

10. $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

12. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Fourier's Model.

13.

Eigenanalysis Facts.

14.

Eigenpair Packages.

15.

The Equation $AP = PD$.

16.

Matrix Eigenanalysis Method.

17.

Basis of Eigenvectors.

18.

Independence of Eigenvectors.

19.

Diagonalization Theory.

20.

Non-diagonalizable Matrices.

21.

Distinct Eigenvalues.

22. $\begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$

23. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

24. $\begin{pmatrix} 2 & 6 & 2 \\ 9 & 3 & 9 \\ 1 & 3 & 1 \end{pmatrix}$

25. $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

26. $\begin{pmatrix} 7 & 12 & 6 \\ 2 & 2 & 2 \\ -7 & -12 & -6 \end{pmatrix}$

27. $\begin{pmatrix} 2 & 2 & -6 \\ -3 & -4 & 3 \\ -3 & -4 & -1 \end{pmatrix}$

Computing 2×2 Eigenpairs.

28.

Computing 2×2 Complex Eigenpairs.

29.

Computing 3×3 Eigenpairs.

30.

Decomposition $A = PDP^{-1}$.

31.

Diagonalization I

32.

Diagonalization II

33.

Non-diagonalizable Matrices

34.

Fourier's Heat Model

35.

9.2 Eigenanalysis II

Discrete Dynamical Systems

The matrix equation

$$\mathbf{y} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}\mathbf{x} \tag{1}$$

predicts the state $\mathbf{y}$ of a system initially in state $\mathbf{x}$ after some fixed elapsed time. The 3×3 matrix A in (1) represents the **dynamics** which changes the state $\mathbf{x}$ into state $\mathbf{y}$. Accordingly, an equation $\mathbf{y} = A\mathbf{x}$ is called a **discrete dynamical system** and A is called a **transition matrix**.

The eigenpairs of A in (1) are shown in *details* page 528 to be $(1, \mathbf{v}_1)$, $(1/2, \mathbf{v}_2)$, $(1/5, \mathbf{v}_3)$ where the eigenvectors are given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 5/4 \\ 13/12 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}. \tag{2}$$

Market Shares. A typical application of discrete dynamical systems is telephone long distance company market shares x_1, x_2, x_3, which are fractions of the total market for long distance service. If three companies provide all the services, then their market fractions add to one: $x_1 + x_2 + x_3 = 1$. The equation $\mathbf{y} = A\mathbf{x}$ gives the market shares of the three companies after a fixed time period, say one year. Then market shares after one, two and three years are given by the **iterates**

$$\begin{aligned} \mathbf{y}_1 &= A\mathbf{x}, \\ \mathbf{y}_2 &= A^2\mathbf{x}, \\ \mathbf{y}_3 &= A^3\mathbf{x}. \end{aligned}$$

Fourier's eigenanalysis model gives succinct and useful formulas for the iterates: if $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$, then

$$\begin{aligned} \mathbf{y}_1 &= A\mathbf{x} &&= a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2 + a_3\lambda_3\mathbf{v}_3, \\ \mathbf{y}_2 &= A^2\mathbf{x} &&= a_1\lambda_1^2\mathbf{v}_1 + a_2\lambda_2^2\mathbf{v}_2 + a_3\lambda_3^2\mathbf{v}_3, \\ \mathbf{y}_3 &= A^3\mathbf{x} &&= a_1\lambda_1^3\mathbf{v}_1 + a_2\lambda_2^3\mathbf{v}_2 + a_3\lambda_3^3\mathbf{v}_3. \end{aligned}$$

The advantage of Fourier's model is that an iterate A^n is computed directly, without computing the powers before it. Because $\lambda_1 = 1$ and $\lim_{n\to\infty} |\lambda_2|^n = \lim_{n\to\infty} |\lambda_3|^n = 0$, then for large n

$$\mathbf{y}_n \approx a_1(1)\mathbf{v}_1 + a_2(0)\mathbf{v}_2 + a_3(0)\mathbf{v}_3 = \begin{pmatrix} a_1 \\ 5a_1/4 \\ 13a_1/12 \end{pmatrix}.$$

The numbers a_1, a_2, a_3 are related to x_1, x_2, x_3 by the equations $a_1 - a_2 - 4a_3 = x_1$, $5a_1/4 + 3a_3 = x_2$, $13a_1/12 + a_2 + a_3 = x_3$. Due to $x_1 + x_2 + x_3 = 1$, the value of a_1 is known, $a_1 = 3/10$. The three market shares after a long time period are therefore predicted to be 3/10, 3/8, 39/120. The reader should verify the identity $\frac{3}{10} + \frac{3}{8} + \frac{39}{120} = 1$.

Stochastic Matrices. The special matrix A in (1) is a **stochastic matrix**, defined by the properties

$$\sum_{i=1}^{n} a_{ij} = 1, \quad a_{kj} \geq 0, \quad k, j = 1, \ldots, n.$$

The definition is memorized by the phrase *each column sum is one.* Stochastic matrices appear in **Leontief input-output models**, popularized by 1973 Nobel Prize economist Wassily Leontief.

Theorem 9 (Stochastic Matrix Properties)
Let A be a stochastic matrix. Then

- **(a)** If $\mathbf{x}$ is a vector with $x_1 + \cdots + x_n = 1$, then $\mathbf{y} = A\mathbf{x}$ satisfies $y_1 + \cdots + y_n = 1$.
- **(b)** If $\mathbf{v}$ is the sum of the columns of I, then $A^T\mathbf{v} = \mathbf{v}$. Therefore, $(1, \mathbf{v})$ is an eigenpair of A^T.
- **(c)** The characteristic equation $\det(A - \lambda I) = 0$ has a root $\lambda = 1$. All other roots satisfy $|\lambda| < 1$.

Proof of Stochastic Matrix Properties:
(a) $\sum_{i=1}^n y_i = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\right) x_j = \sum_{j=1}^n (1)x_j = 1$.

(b) Entry j of $A^T\mathbf{v}$ is given by the sum $\sum_{i=1}^n a_{ij} = 1$.

(c) Apply **(b)** and the determinant rule $\det(B^T) = \det(B)$ with $B = A - \lambda I$ to obtain eigenvalue 1. Any other root λ of the characteristic equation has a corresponding eigenvector $\mathbf{x}$ satisfying $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Let index j be selected such that $M = |x_j| > 0$ has largest magnitude. Then $\sum_{i \neq j} a_{ij}x_j + (a_{jj} - \lambda)x_j = 0$ implies $\lambda = \sum_{i=1}^n a_{ij}\dfrac{x_j}{M}$. Because $\sum_{i=1}^n a_{ij} = 1$, λ is a convex combination of n complex numbers $\{x_j/M\}_{j=1}^n$. These complex numbers are located in the unit disk, a convex set, therefore λ is located in the unit disk. By induction on n, motivated by the geometry for $n = 2$, it is argued that $|\lambda| = 1$ cannot happen for λ an eigenvalue different from 1 (details left to the reader). Therefore, $|\lambda| < 1$.

Details for the eigenpairs of (1): To be computed are the eigenvalues and eigenvectors for the 3×3 matrix

$$A = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}.$$

Eigenvalues. The roots $\lambda = 1, 1/2, 1/5$ of the characteristic equation $\det(A - \lambda I) = 0$ are found by these details:

$$\begin{aligned}
0 &= \det(A-\lambda I) \\
&= \begin{vmatrix} .5-\lambda & .4 & 0 \\ .3 & .5-\lambda & .3 \\ .2 & .1 & .7-\lambda \end{vmatrix} \\
&= \frac{1}{10} - \frac{8}{10}\lambda + \frac{17}{10}\lambda^2 - \lambda^3 && \text{Expand by cofactors.} \\
&= -\frac{1}{10}(\lambda-1)(2\lambda-1)(5\lambda-1) && \text{Factor the cubic.}
\end{aligned}$$

The factorization was found by long division of the cubic by $\lambda - 1$, the idea born from the fact that 1 is a root and therefore $\lambda - 1$ is a factor (the Factor Theorem of college algebra). An answer check in `maple`:

```
with(linalg):
A:=(1/10)*matrix([[5,4,0],[3,5,3],[2,1,7]]);
B:=evalm(A-lambda*diag(1,1,1));
eigenvals(A); factor(det(B));
```

Eigenpairs. To each eigenvalue $\lambda = 1, 1/2, 1/5$ corresponds one **rref** calculation, to find the eigenvectors paired to λ. The three eigenvectors are given by (2). The details:

Eigenvalue $\lambda = 1$.

$$\begin{aligned}
A-(1)I &= \begin{pmatrix} .5-1 & .4 & 0 \\ .3 & .5-1 & .3 \\ .2 & .1 & .7-1 \end{pmatrix} \\
&\approx \begin{pmatrix} -5 & 4 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Multiply rule, multiplier=10.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule twice.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 0 & 13 & -15 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \end{pmatrix} && \text{Multiply rule and combination rule.} \\
&\approx \begin{pmatrix} 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathbf{rref}(A-(1)I)
\end{aligned}$$

An equivalent reduced echelon system is $x - 12z/13 = 0$, $y - 15z/13 = 0$. The free variable assignment is $z = t_1$ and then $x = 12t_1/13$, $y = 15t_1/13$. Let $x = 1$; then $t_1 = 13/12$. An eigenvector is given by $x = 1$, $y = 4/5$, $z = 13/12$.

Eigenvalue $\lambda = 1/2$.

$$
\begin{aligned}
A-(1/2)I &= \begin{pmatrix} .5-.5 & .4 & 0 \\ .3 & .5-.5 & .3 \\ .2 & .1 & .7-.5 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 4 & 0 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{pmatrix} && \text{Multiply rule, factor=10.} \\
&\approx \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply rules.} \\
&= \mathbf{rref}(A-.5I)
\end{aligned}
$$

An eigenvector is found from the equivalent reduced echelon system $y = 0$, $x + z = 0$ to be $x = -1$, $y = 0$, $z = 1$.

Eigenvalue $\lambda = 1/5$.

$$
\begin{aligned}
A-(1/5)I &= \begin{pmatrix} .5-.2 & .4 & 0 \\ .3 & .5-.2 & .3 \\ .2 & .1 & .7-.2 \end{pmatrix} \\
&\approx \begin{pmatrix} 3 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 5 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination rule.} \\
&= \mathbf{rref}(A-(1/5)I)
\end{aligned}
$$

An eigenvector is found from the equivalent reduced echelon system $x+4z = 0$, $y - 3z = 0$ to be $x = -4$, $y = 3$, $z = 1$.

An answer check in `maple`:

```
with(linalg):
A:=(1/10)*matrix([[5,4,0],[3,5,3],[2,1,7]]);
eigenvects(A);
```

Coupled and Uncoupled Systems

The linear system of differential equations

$$
\begin{aligned}
x_1' &= -x_1 - x_3, \\
x_2' &= 4x_1 - x_2 - 3x_3, \\
x_3' &= 2x_1 - 4x_3,
\end{aligned} \tag{3}
$$

is called **coupled**, whereas the linear system of growth-decay equations

$$
\begin{aligned}
y_1' &= -3y_1, \\
y_2' &= -y_2, \\
y_3' &= -2y_3,
\end{aligned} \tag{4}
$$

is called **uncoupled**. The terminology *uncoupled* means that each differential equation in system (4) depends on exactly one variable, e.g., $y_1' = -3y_1$ depends only on variable y_1. In a *coupled* system, one of the differential equations must involve two or more variables.

Matrix characterization. Coupled system (3) and uncoupled system (4) can be written in matrix form, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{y}' = D\mathbf{y}$, with coefficient matrices

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If the coefficient matrix is **diagonal**, then the system is **uncoupled**. If the coefficient matrix is **not diagonal**, then one of the corresponding differential equations involves two or more variables and the system is called **coupled** or **cross-coupled**.

Solving Uncoupled Systems

An uncoupled system consists of independent growth-decay equations of the form $u' = au$. The solution formula $u = ce^{at}$ then leads to the general solution of the system of equations. For instance, system (4) has general solution

$$\begin{aligned} y_1 &= c_1 e^{-3t}, \\ y_2 &= c_2 e^{-t}, \\ y_3 &= c_3 e^{-2t}, \end{aligned} \tag{5}$$

where c_1, c_2, c_3 are **arbitrary constants**. The number of constants equals the dimension of the diagonal matrix D.

Coordinates and Coordinate Systems

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three independent vectors in $\mathcal{R}^3$, then the matrix

$$P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

is invertible. The columns $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of P are called a **coordinate system**. The matrix P is called a **change of coordinates**.

Every vector $\mathbf{v}$ in $\mathcal{R}^3$ can be uniquely expressed as

$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3.$$

The values t_1, t_2, t_3 are called the **coordinates** of $\mathbf{v}$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, or more succinctly, the **coordinates of $\mathbf{v}$ relative to P**.

Viewpoint of a Driver

The physical meaning of a coordinate system $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ can be understood by considering an auto going up a mountain road. Choose orthogonal $\mathbf{v}_1$ and $\mathbf{v}_2$ to give positions in the driver's seat and define $\mathbf{v}_3$ be the seat-back direction. These are **local coordinates** as viewed from the driver's seat. The road map coordinates x, y and the altitude z define the **global coordinates** for the auto's position $\mathbf{p} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

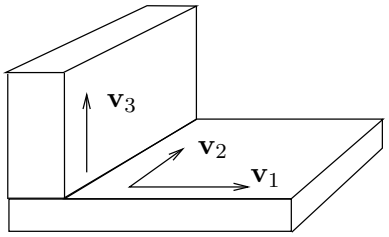


Figure 1. An auto seat. The vectors $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$, $\mathbf{v}_3(t)$ form an orthogonal triad which is a local coordinate system from the driver's viewpoint. The orthogonal triad changes continuously in t.

Change of Coordinates

A coordinate change from $\mathbf{y}$ to $\mathbf{x}$ is a linear algebraic equation $\mathbf{x} = P\mathbf{y}$ where the $n \times n$ matrix P is required to be invertible ($\det(P) \neq 0$). To illustrate, an instance of a change of coordinates from $\mathbf{y}$ to $\mathbf{x}$ is given by the linear equations

$$\mathbf{x} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \mathbf{y} \quad \text{or} \quad \begin{cases} x_1 &= y_1 + y_3, \\ x_2 &= y_1 + y_2 - y_3, \\ x_3 &= 2y_1 + y_3. \end{cases} \tag{6}$$

Constructing Coupled Systems

A general method exists to construct rich examples of coupled systems. The idea is to substitute a change of variables into a given uncoupled system. Consider a diagonal system $\mathbf{y}' = D\mathbf{y}$, like (4), and a change of variables $\mathbf{x} = P\mathbf{y}$, like (6). Differential calculus applies to give

$$\begin{aligned} \mathbf{x}' &= (P\mathbf{y})' \\ &= P\mathbf{y}' \\ &= PD\mathbf{y} \\ &= PDP^{-1}\,\mathbf{x}. \end{aligned} \tag{7}$$

The matrix $A = PDP^{-1}$ is *not triangular* in general, and therefore the change of variables produces a **cross-coupled** system.

An illustration. To give an example, substitute into uncoupled system (4) the change of variable equations (6). Use equation (7) to obtain

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \mathbf{x} \quad \text{or} \quad \begin{cases} x_1' = -x_1 - x_3, \\ x_2' = 4x_1 - x_2 - 3x_3, \\ x_3' = 2x_1 - 4x_3. \end{cases} \tag{8}$$

This **cross-coupled** system (8) can be solved using relations (6), (5) and $\mathbf{x} = P\mathbf{y}$ to give the general solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{-t} \\ c_3 e^{-2t} \end{pmatrix}. \tag{9}$$

Changing Coupled Systems to Uncoupled

We ask this question, motivated by the above calculations:

> Can every coupled system $\mathbf{x}'(t) = A\mathbf{x}(t)$ be subjected to a change of variables $\mathbf{x} = P\mathbf{y}$ which converts the system into a completely uncoupled system for variable $\mathbf{y}(t)$?

Under certain circumstances, this is true, and it leads to an elegant and especially simple expression for the general solution of the differential system, as in (9):

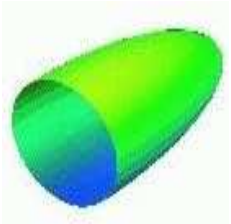
$$\mathbf{x}(t) = P\mathbf{y}(t).$$

The **task of eigenanalysis** is to simultaneously calculate from a cross-coupled system $\mathbf{x}' = A\mathbf{x}$ the change of variables $\mathbf{x} = P\mathbf{y}$ and the diagonal matrix D in the uncoupled system $\mathbf{y}' = D\mathbf{y}$

The **eigenanalysis coordinate system** is the set of n independent vectors extracted from the columns of P. In this coordinate system, the cross-coupled differential system (3) simplifies into a system of uncoupled growth-decay equations (4). Hence the terminology, *the method of simplifying coordinates*.

Eigenanalysis and Footballs

An ellipsoid or *football* is a geometric object described by its **semi-axes** (see Figure 2). In the vector representation, the **semi-axis directions** are unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and the **semi-axis lengths** are the constants a, b, c. The vectors $a\mathbf{v}_1$, $b\mathbf{v}_2$, $c\mathbf{v}_3$ form an **orthogonal triad**.



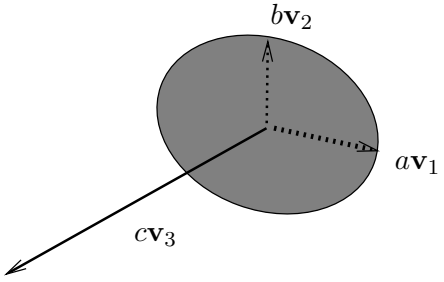


Figure 2. A football. An ellipsoid is built from orthonormal semi-axis directions $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and the semi-axis lengths a, b, c. The semi-axis vectors are $a\mathbf{v}_1$, $b\mathbf{v}_2$, $c\mathbf{v}_3$.

Two vectors **a**, **b** are *orthogonal* if both are nonzero and their dot product $\mathbf{a} \cdot \mathbf{b}$ is zero. Vectors are **orthonormal** if each has unit length and they are pairwise orthogonal. The orthogonal triad is an **invariant** of the ellipsoid's algebraic representations. Algebra does not change the triad: the invariants $a\mathbf{v}_1$, $b\mathbf{v}_2$, $c\mathbf{v}_3$ must somehow be **hidden** in the equations that represent the football.

Algebraic eigenanalysis finds the hidden invariant triad $a\mathbf{v}_1$, $b\mathbf{v}_2$, $c\mathbf{v}_3$ from the ellipsoid's algebraic equations. Suppose, for instance, that the equation of the ellipsoid is supplied as

$$x^2 + 4y^2 + xy + 4z^2 = 16.$$

A symmetric matrix A is constructed in order to write the equation in the form $\mathbf{X}^T A \mathbf{X} = 16$, where $\mathbf{X}$ has components x, y, z. The replacement equation is[4]

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 16. \tag{10}$$

It is the 3×3 symmetric matrix A in (10) that is subjected to algebraic eigenanalysis. The matrix calculation will compute the unit semi-axis directions $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, called the **hidden vectors** or **eigenvectors**. The semi-axis lengths a, b, c are computed at the same time, by finding the **hidden values**[5] or **eigenvalues** λ_1, λ_2, λ_3, known to satisfy the relations

$$\lambda_1 = \frac{16}{a^2}, \quad \lambda_2 = \frac{16}{b^2}, \quad \lambda_3 = \frac{16}{c^2}\,.$$

For the illustration, the football dimensions are $a = 2$, $b = 1.98$, $c = 4.17$. Details of the computation are delayed until page 536.

The Ellipse and Eigenanalysis

An ellipse equation in **standard form** is $\lambda_1 x^2 + \lambda_2 y^2 = 1$, where $\lambda_1 = 1/a^2$, $\lambda_2 = 1/b^2$ are expressed in terms of the semi-axis lengths a, b. The expression $\lambda_1 x^2 + \lambda_2 y^2$ is called a **quadratic form**. The study of the ellipse $\lambda_1 x^2 + \lambda_2 y^2 = 1$ is equivalent to the study of the quadratic form equation

$$\mathbf{r}^T D \mathbf{r} = 1, \quad \text{where} \quad \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

[4] The reader should pause here and multiply matrices in order to verify this statement. Halving of the entries corresponding to cross-terms generalizes to any ellipsoid.

[5] The terminology *hidden* arises because neither the semi-axis lengths nor the semi-axis directions are revealed directly by the ellipsoid equation.

Cross-terms. An ellipse may be represented by an equation in a uv-coordinate system having a cross-term uv, e.g., $4u^2+8uv+10v^2 = 5$. The expression $4u^2 + 8uv + 10v^2$ is again called a quadratic form. Calculus courses provide methods to eliminate the cross-term and represent the equation in standard form, by a **rotation**

$$\begin{pmatrix} u \\ v \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The angle θ in the rotation matrix R represents the rotation of uv-coordinates into standard xy-coordinates.

Eigenanalysis computes angle θ through the columns of R, which are the unit semi-axis directions $\mathbf{v}_1$, $\mathbf{v}_2$ for the ellipse $4u^2 + 8uv + 10v^2 = 5$. If the quadratic form $4u^2 + 8uv + 10v^2$ is represented as $\mathbf{r}^T A\,\mathbf{r}$, then

$$\mathbf{r} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}, \quad R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

$$\lambda_1 = 12, \quad \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 2, \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Rotation matrix angle θ. The components of eigenvector $\mathbf{v}_1$ can be used to determine $\theta = -63.4°$:

$$\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{cases} \cos\theta &= \frac{1}{\sqrt{5}}, \\ -\sin\theta &= \frac{2}{\sqrt{5}}. \end{cases}$$

The interpretation of angle θ: rotate the orthonormal basis $\mathbf{v}_1$, $\mathbf{v}_2$ by angle $\theta = -63.4°$ in order to obtain the standard unit basis vectors $\mathbf{i}$, $\mathbf{j}$. Most calculus texts discuss only the inverse rotation, where x, y are given in terms of u, v. In these references, θ is the negative of the value given here, due to a different geometric viewpoint.[6]

Semi-axis lengths. The lengths $a \approx 1.55$, $b \approx 0.63$ for the ellipse $4u^2+8uv+10v^2 = 5$ are computed from the eigenvalues $\lambda_1 = 12$, $\lambda_2 = 2$ of matrix A by the equations

$$\frac{\lambda_1}{5} = \frac{1}{a^2}, \quad \frac{\lambda_2}{5} = \frac{1}{b^2}.$$

Geometry. The ellipse $4u^2 + 8uv + 10v^2 = 5$ is completely determined by the orthogonal semi-axis vectors $a\mathbf{v}_1$, $b\mathbf{v}_2$. The rotation R is a rigid motion which maps these vectors into $a\vec{\imath}$, $b\vec{\jmath}$, where $\vec{\imath}$ and $\vec{\jmath}$ are the standard unit vectors in the plane.

The θ-rotation R maps $4u^2+8uv+10v^2 = 5$ into the xy-equation $\lambda_1 x^2 + \lambda_2 y^2 = 5$, where λ_1, λ_2 are the eigenvalues of A. To see why, let $\mathbf{r} = R\mathbf{s}$ where $\mathbf{s} = \begin{pmatrix} x & y \end{pmatrix}^T$. Then $\mathbf{r}^T A\mathbf{r} = \mathbf{s}^T(R^T AR)\mathbf{s}$. Using $R^T R = I$ gives $R^{-1} = R^T$ and $R^T AR = \mathbf{diag}(\lambda_1, \lambda_2)$. Finally, $\mathbf{r}^T A\mathbf{r} = \lambda_1 x^2 + \lambda_2 y^2$.

[6] Rod Serling, author of *The Twilight Zone*, enjoyed the view from the other side of the mirror.

Orthogonal Triad Computation

Let's compute the semiaxis directions $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for the ellipsoid $x^2 + 4y^2 + xy + 4z^2 = 16$. To be applied is Theorem 7. As explained on page 534, the starting point is to represent the ellipsoid equation as a quadratic form $X^TAX = 16$, where the symmetric matrix A is defined by

$$A = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

College algebra. The **characteristic polynomial** $\det(A - \lambda I) = 0$ determines the eigenvalues or hidden values of the matrix A. By cofactor expansion, this polynomial equation is

$$(4 - \lambda)((1 - \lambda)(4 - \lambda) - 1/4) = 0$$

with roots 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

Eigenpairs. It will be shown that three eigenpairs are

$$\lambda_1 = 4, \quad \mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_2 = \frac{5 + \sqrt{10}}{2}, \quad \mathbf{x}_2 = \begin{pmatrix} \sqrt{10} - 3 \\ 1 \\ 0 \end{pmatrix},$$

$$\lambda_3 = \frac{5 - \sqrt{10}}{2}, \quad \mathbf{x}_3 = \begin{pmatrix} \sqrt{10} + 3 \\ -1 \\ 0 \end{pmatrix}.$$

The vector norms of the eigenvectors are given by $\|\mathbf{x}_1\| = 1$, $\|\mathbf{x}_2\| = \sqrt{20 + 6\sqrt{10}}$, $\|\mathbf{x}_3\| = \sqrt{20 - 6\sqrt{10}}$. The orthonormal semi-axis directions $\mathbf{v}_k = \mathbf{x}_k / \|\mathbf{x}_k\|$, $k = 1, 2, 3$, are then given by the formulas

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \frac{\sqrt{10}-3}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20-6\sqrt{10}}} \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} \frac{\sqrt{10}+3}{\sqrt{20+6\sqrt{10}}} \\ \frac{-1}{\sqrt{20+6\sqrt{10}}} \\ 0 \end{pmatrix}.$$

Frame sequence details.

$$\mathbf{aug}(A - \lambda_1 I, \mathbf{0}) = \left(\begin{array}{ccc|c} 1-4 & 1/2 & 0 & 0 \\ 1/2 & 4-4 & 0 & 0 \\ 0 & 0 & 4-4 & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Used combination, multiply and swap rules. Found **rref**.

$$\mathbf{aug}(A-\lambda_2 I,\mathbf{0}) = \left(\begin{array}{ccc|c} \frac{-3-\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3-\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3-\sqrt{10}}{2} & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 3-\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{All three rules.}$$

$$\mathbf{aug}(A-\lambda_3 I,\mathbf{0}) = \left(\begin{array}{ccc|c} \frac{-3+\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3+\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{10}}{2} & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 3+\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{All three rules.}$$

Solving the corresponding reduced echelon systems gives the preceding formulas for the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. The equation for the ellipsoid is $\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 16$, where the multipliers of the square terms are the eigenvalues of A and X, Y, Z define the new coordinate system determined by the eigenvectors of A. This equation can be re-written in the form $X^2/a^2 + Y^2/b^2 + Z^2/c^2 = 1$, provided the semi-axis lengths a, b, c are defined by the relations $a^2 = 16/\lambda_1$, $b^2 = 16/\lambda_2$, $c^2 = 16/\lambda_3$. After computation, $a = 2$, $b = 1.98$, $c = 4.17$.

9.3 Advanced Topics in Linear Algebra

Diagonalization and Jordan's Theorem

A system of differential equations $\mathbf{x}' = A\mathbf{x}$ can be transformed to an uncoupled system $\mathbf{y}' = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)\mathbf{y}$ by a change of variables $\mathbf{x} = P\mathbf{y}$, provided P is invertible and A satisfies the relation

$$AP = P\,\mathbf{diag}(\lambda_1, \ldots, \lambda_n). \tag{1}$$

A matrix A is said to be **diagonalizable** provided (1) holds. This equation is equivalent to the system of equations $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, $k = 1, \ldots, n$, where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the columns of matrix P. Since P is assumed invertible, each of its columns are nonzero, and therefore $(\lambda_k, \mathbf{v}_k)$ is an eigenpair of A, $1 \le k \le n$. The values λ_k need not be distinct (e.g., all $\lambda_k = 1$ if A is the identity). This proves:

Theorem 10 (Diagonalization)
An $n \times n$ matrix A is diagonalizable if and only if A has n eigenpairs $(\lambda_k, \mathbf{v}_k)$, $1 \le k \le n$, with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ independent. In this case,

$$A = PDP^{-1}$$

where $D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$ and the matrix P has columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Theorem 11 (Jordan's theorem)
Any $n \times n$ matrix A can be represented in the form

$$A = PTP^{-1}$$

where P is invertible and T is upper triangular. The diagonal entries of T are eigenvalues of A.

Proof: We proceed by induction on the dimension n of A. For $n = 1$ there is nothing to prove. Assume the result for dimension n, and let's prove it when A is $(n+1)\times(n+1)$. Choose an eigenpair $(\lambda_1, \mathbf{v}_1)$ of A with $\mathbf{v}_1 \neq \mathbf{0}$. Complete a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ for $\mathcal{R}^{n+1}$ and define $V = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$. Then $V^{-1}AV = \left(\begin{array}{c|c} \lambda_1 & B \\ \hline \mathbf{0} & A_1 \end{array}\right)$ for some matrices B and A_1. The induction hypothesis implies there is an invertible $n \times n$ matrix P_1 and an upper triangular matrix T_1 such that $A_1 = P_1 T_1 P_1^{-1}$. Let $R = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_1 \end{array}\right)$ and $\mathbf{T} = \left(\begin{array}{c|c} \lambda_1 & BT_1 \\ \hline 0 & T_1 \end{array}\right)$. Then T is upper triangular and $(V^{-1}AV)R = RT$, which implies $A = PTP^{-1}$ for $P = VR$. The induction is complete.

Cayley-Hamilton Identity

A celebrated and deep result for powers of matrices was discovered by Cayley and Hamilton (see [**?**]), which says that an $n \times n$ matrix A satisfies its own characteristic equation. More precisely:

Theorem 12 (Cayley-Hamilton)
Let $\det(A - \lambda I)$ be expanded as the nth degree polynomial

$$p(\lambda) = \sum_{j=0}^{n} c_j \lambda^j,$$

for some coefficients c_0, ..., c_{n-1} and $c_n = (-1)^n$. Then A satisfies the equation $p(\lambda) = 0$, that is,

$$p(A) \equiv \sum_{j=0}^{n} c_j A^j = 0.$$

In factored form in terms of the eigenvalues $\{\lambda_j\}_{j=1}^{n}$ (duplicates possible), the matrix equation $p(A) = 0$ can be written as

$$(-1)^n (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = 0.$$

Proof: If A is diagonalizable, $AP = P\,\mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, then the proof is obtained from the simple expansion

$$A^j = P\,\mathbf{diag}(\lambda_1^j, \ldots, \lambda_n^j) P^{-1},$$

because summing across this identity leads to

$$\begin{array}{rcl} p(A) & = & \sum_{j=0}^{n} c_j A^j \\ & = & P \left(\sum_{j=0}^{n} c_j\, \mathbf{diag}(\lambda_1^j, \ldots, \lambda_n^j) \right) P^{-1} \\ & = & P\,\mathbf{diag}(p(\lambda_1), \ldots, p(\lambda_n)) P^{-1} \\ & = & P\,\mathbf{diag}(0, \ldots, 0) P^{-1} \\ & = & 0. \end{array}$$

If A is not diagonalizable, then this proof fails. To handle the general case, we apply **Jordan's theorem**, which says that $A = PTP^{-1}$ where T is *upper triangular* (instead of *diagonal*) and the not necessarily distinct eigenvalues λ_1, ..., λ_n of A appear on the diagonal of T. Using Jordan's theorem, define

$$A_\epsilon = P(T + \epsilon\,\mathbf{diag}(1, 2, \ldots, n)) P^{-1}.$$

For small $\epsilon > 0$, the matrix A_ϵ has distinct eigenvalues $\lambda_j + \epsilon j$, $1 \le j \le n$. Then the diagonalizable case implies that A_ϵ satisfies its characteristic equation. Let $p_\epsilon(\lambda) = \det(A_\epsilon - \lambda I)$. Use $0 = \lim_{\epsilon \to 0} p_\epsilon(A_\epsilon) = p(A)$ to complete the proof.

An Extension of Jordan's Theorem

Theorem 13 (Jordan's Extension)
Any $n \times n$ matrix A can be represented in the block triangular form

$$A = PTP^{-1}, \quad T = \mathbf{diag}(T_1, \ldots, T_k),$$

where P is invertible and each matrix T_i is upper triangular with diagonal entries equal to a single eigenvalue of A.

The proof of the theorem is based upon Jordan's theorem, and proceeds by induction. The reader is invited to try to find a proof, or read further in the text, where this theorem is presented as a special case of the Jordan decomposition.

Solving Block Triangular Differential Systems

A matrix differential system $\mathbf{y}'(t) = T\mathbf{y}(t)$ with T block upper triangular splits into scalar equations which can be solved by elementary methods for first order scalar differential equations. To illustrate, consider the system

$$\begin{aligned} y_1' &= 3y_1 + x_2 + y_3, \\ y_2' &= 3y_2 + y_3, \\ y_3' &= 2y_3. \end{aligned}$$

The techniques that apply are the growth-decay formula for $u' = ku$ and the integrating factor method for $u' = ku + p(t)$. Working backwards from the last equation, using back-substitution, gives

$$\begin{aligned} y_3 &= c_3 e^{2t}, \\ y_2 &= c_2 e^{3t} - c_3 e^{2t}, \\ y_1 &= (c_1 + c_2 t) e^{3t}. \end{aligned}$$

What has been said here applies to any triangular system $\mathbf{y}'(t) = T\mathbf{y}(t)$, in order to write an exact formula for the solution $\mathbf{y}(t)$.

If A is an $n \times n$ matrix, then Jordan's theorem gives $A = PTP^{-1}$ with T block upper triangular and P invertible. The change of variable $\mathbf{x}(t) = P\mathbf{y}(t)$ changes $\mathbf{x}'(t) = A\mathbf{x}(t)$ into the block triangular system $\mathbf{y}'(t) = T\mathbf{y}(t)$.

There is no special condition on A, to effect the change of variable $\mathbf{x}(t) = P\mathbf{y}(t)$. The solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is a product of the invertible matrix P and a column vector $\mathbf{y}(t)$; the latter is the solution of the block triangular system $\mathbf{y}'(t) = T\mathbf{y}(t)$, obtained by growth-decay and integrating factor methods.

The *importance of this idea* is to provide a solid method for solving any system $\mathbf{x}'(t) = A\mathbf{x}(t)$. In later sections, we outline how to find

the matrix P and the matrix T, in Jordan's extension $A = PTP^{-1}$. The additional theory provides efficient matrix methods for solving any system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Symmetric Matrices and Orthogonality

Described here is a process due to Gram-Schmidt for replacing a given set of independent eigenvectors by another set of eigenvectors which are of unit length and **orthogonal** (dot product zero or 90 degrees apart). The process, which applies to any independent set of vectors, is especially useful in the case of eigenanalysis of a **symmetric** matrix: $A^T = A$.

Unit eigenvectors. An eigenpair $(\lambda, \mathbf{x})$ of A can always be selected so that $\|\mathbf{x}\| = 1$. If $\|\mathbf{x}\| \neq 1$, then replace eigenvector $\mathbf{x}$ by the scalar multiple $c\mathbf{x}$, where $c = 1/\|\mathbf{x}\|$. By this small change, it can be assumed that the eigenvector has unit length. If in addition the eigenvectors are *orthogonal*, then the eigenvectors are said to be **orthonormal**.

Theorem 14 (Orthogonality of Eigenvectors)
Assume that $n \times n$ matrix A is **symmetric**, $A^T = A$. If $(\alpha, \mathbf{x})$ and $(\beta, \mathbf{y})$ are eigenpairs of A with $\alpha \neq \beta$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal: $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof: To prove this result, compute $\alpha\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} = \mathbf{x}^T A\mathbf{y}$. Also, $\beta\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T A\mathbf{y}$. Subtracting the relations implies $(\alpha - \beta)\mathbf{x} \cdot \mathbf{y} = 0$, giving $\mathbf{x} \cdot \mathbf{y} = 0$ due to $\alpha \neq \beta$. The proof is complete.

Theorem 15 (Real Eigenvalues)
If $A^T = A$, then all eigenvalues of A are real. Consequently, matrix A has n real eigenvalues counted according to multiplicity.

Proof: The second statement is due to the fundamental theorem of algebra. To prove the eigenvalues are real, it suffices to prove $\lambda = \overline{\lambda}$ when $A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$. We admit that $\mathbf{v}$ may have complex entries. We will use $\overline{A} = A$ (A is real). Take the complex conjugate across $A\mathbf{v} = \lambda\mathbf{v}$ to obtain $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. Transpose $A\mathbf{v} = \lambda\mathbf{v}$ to obtain $\mathbf{v}^T A^T = \lambda\mathbf{v}^T$ and then conclude $\mathbf{v}^T A = \lambda\mathbf{v}^T$ from $A^T = A$. Multiply this equation by $\overline{\mathbf{v}}$ on the right to obtain $\mathbf{v}^T A\overline{\mathbf{v}} = \lambda\mathbf{v}^T\overline{\mathbf{v}}$. Then multiply $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$ by $\mathbf{v}^T$ on the left to obtain $\mathbf{v}^T A\overline{\mathbf{v}} = \overline{\lambda}\mathbf{v}^T\overline{\mathbf{v}}$. Then we have

$$\lambda\mathbf{v}^T\overline{\mathbf{v}} = \overline{\lambda}\mathbf{v}^T\overline{\mathbf{v}}.$$

Because $\mathbf{v}^T\overline{\mathbf{v}} = \sum_{j=1}^{n} |v_j|^2 > 0$, then $\lambda = \overline{\lambda}$ and λ is real. The proof is complete.

Theorem 16 (Independence of Orthogonal Sets)
Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a set of nonzero orthogonal vectors. Then this set is independent.

Proof: Form the equation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, the plan being to solve for $c_1, \ldots, c_k$. Take the dot product of the equation with $\mathbf{v}_1$. Then

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + \cdots + c_k\mathbf{v}_1 \cdot \mathbf{v}_k = \mathbf{v}_1 \cdot \mathbf{0}.$$

All terms on the left side except one are zero, and the right side is zero also, leaving the relation

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_1 = 0.$$

Because $\mathbf{v}_1$ is not zero, then $c_1 = 0$. The process can be applied to the remaining coefficients, resulting in

$$c_1 = c_2 = \cdots = c_k = 0,$$

which proves independence of the vectors.

The Gram-Schmidt process

The eigenvectors of a symmetric matrix A may be constructed to be orthogonal. First of all, observe that eigenvectors corresponding to distinct eigenvalues are orthogonal by Theorem 14. It remains to construct from k independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$, corresponding to a single eigenvalue λ, another set of independent eigenvectors $\mathbf{y}_1, \ldots, \mathbf{y}_k$ for λ which are pairwise orthogonal. The idea, due to Gram-Schmidt, applies to any set of k independent vectors.

Application of the Gram-Schmidt process can be illustrated by example: let $(-1, \mathbf{v}_1)$, $(2, \mathbf{v}_2)$, $(2, \mathbf{v}_3)$, $(2, \mathbf{v}_4)$ be eigenpairs of a 4×4 symmetric matrix A. Then $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. The vectors $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ belong to eigenvalue $\lambda = 2$, but they are not necessarily orthogonal. The Gram-Schmidt process replaces these vectors by $\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_4$ which are pairwise orthogonal. The result is that eigenvectors $\mathbf{v}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_4$ are pairwise orthogonal.

Theorem 17 (Gram-Schmidt)
Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be independent n-vectors. The set of vectors $\mathbf{y}_1, \ldots, \mathbf{y}_k$ constructed below as linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are pairwise orthogonal and independent.

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}_1 \\
\mathbf{y}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 \\
\mathbf{y}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_2}{\mathbf{y}_2 \cdot \mathbf{y}_2}\mathbf{y}_2 \\
&\vdots \\
\mathbf{y}_k &= \mathbf{x}_k - \frac{\mathbf{x}_k \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{y}_{k-1}}{\mathbf{y}_{k-1} \cdot \mathbf{y}_{k-1}}\mathbf{y}_{k-1}
\end{aligned}$$

Proof: Induction will be applied on k to show that $\mathbf{y}_1, \ldots, \mathbf{y}_k$ are nonzero and orthogonal. If $k = 1$, then there is just one nonzero vector constructed $\mathbf{y}_1 = \mathbf{x}_1$. Orthogonality for $k = 1$ is not discussed because there are no pairs to test. Assume the result holds for $k-1$ vectors. Let's verify that it holds for k vectors, $k > 1$. Assume orthogonality $\mathbf{y}_i \cdot \mathbf{y}_j = 0$ and $\mathbf{y}_i \neq \mathbf{0}$ for $1 \leq i, j \leq k-1$. It remains to test $\mathbf{y}_i \cdot \mathbf{y}_k = 0$ for $1 \leq i \leq k-1$ and $\mathbf{y}_k \neq \mathbf{0}$. The test depends upon the identity

$$\mathbf{y}_i \cdot \mathbf{y}_k = \mathbf{y}_i \cdot \mathbf{x}_k - \sum_{j=1}^{k-1} \frac{\mathbf{x}_k \cdot \mathbf{y}_j}{\mathbf{y}_j \cdot \mathbf{y}_j} \mathbf{y}_i \cdot \mathbf{y}_j,$$

which is obtained from the formula for $\mathbf{y}_k$ by taking the dot product with $\mathbf{y}_i$. In the identity, $\mathbf{y}_i \cdot \mathbf{y}_j = 0$ by the induction hypothesis for $1 \leq j \leq k-1$ and $j \neq i$. Therefore, the summation in the identity contains just the term for index $j = i$, and the contribution is $\mathbf{y}_i \cdot \mathbf{x}_k$. This contribution cancels the leading term on the right in the identity, resulting in the orthogonality relation $\mathbf{y}_i \cdot \mathbf{y}_k = 0$. If $\mathbf{y}_k = \mathbf{0}$, then $\mathbf{x}_k$ is a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_{k-1}$. But each $\mathbf{y}_j$ is a linear combination of $\{\mathbf{x}_i\}_{i=1}^{j}$, therefore $\mathbf{y}_k = \mathbf{0}$ implies $\mathbf{x}_k$ is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$, a contradiction to the independence of $\{\mathbf{x}_i\}_{i=1}^{k}$. The proof is complete.

Orthogonal Projection

Reproduced here is the basic material on shadow projection, for the convenience of the reader. The ideas are then extended to obtain the orthogonal projection onto a subspace V of $\mathcal{R}^n$. Finally, the orthogonal projection formula is related to the Gram-Schmidt equations.

The **shadow projection** of vector $\vec{X}$ onto the direction of vector $\vec{Y}$ is the number d defined by

$$d = \frac{\vec{X} \cdot \vec{Y}}{|\vec{Y}|}.$$

The triangle determined by $\vec{X}$ and $d\dfrac{\vec{Y}}{|\vec{Y}|}$ is a right triangle.

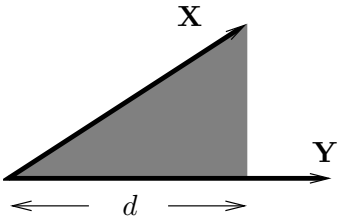


Figure 3. Shadow projection d of vector X onto the direction of vector Y.

The **vector shadow projection** of $\vec{X}$ onto the line L through the origin in the direction of $\vec{Y}$ is defined by

$$\mathbf{proj}_{\vec{Y}}(\vec{X}) = d\frac{\vec{Y}}{|\vec{Y}|} = \frac{\vec{X} \cdot \vec{Y}}{\vec{Y} \cdot \vec{Y}}\vec{Y}.$$

Orthogonal projection for dimension 1. The extension of the shadow projection formula to a subspace V of $\mathcal{R}^n$ begins with unitizing $\vec{Y}$ to isolate the vector direction $\mathbf{u} = \vec{Y}/\|\vec{Y}\|$ of line L. Define the subspace $V = \mathbf{span}\{\mathbf{u}\}$. Then V is identical to L. We define the **orthogonal projection** by the formula

$$\mathbf{Proj}_V(\mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}, \quad V = \mathbf{span}\{\mathbf{u}\}.$$

The reader is asked to verify that

$$\mathbf{proj}_{\vec{Y}}(\mathbf{x}) = d\mathbf{u} = \mathbf{Proj}_V(\mathbf{x}).$$

These equalities mean that the orthogonal projection is the vector shadow projection when V is one dimensional.

Orthogonal projection for dimension k. Consider a subspace V of $\mathcal{R}^n$ given as the span of orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Define the **orthogonal projection** by the formula

$$\mathbf{Proj}_V(\mathbf{x}) = \sum_{j=1}^{k} (\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j, \quad V = \mathbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}.$$

Orthogonal projection and Gram-Schmidt. Define $\mathbf{y}_1, \ldots, \mathbf{y}_k$ by the Gram-Schmidt relations on page 542. Let $\mathbf{u}_j = \mathbf{y}_j/\|\mathbf{y}_j\|$ for $j = 1, \ldots, k$. Then $V_{j-1} = \mathbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}\}$ is a subspace of $\mathcal{R}^n$ of dimension $j-1$ with orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}$ and

$$\begin{aligned} \mathbf{y}_j &= \mathbf{x}_j - \frac{\mathbf{x}_j \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 - \cdots - \frac{\mathbf{x}_k \cdot \mathbf{y}_{j-1}}{\mathbf{y}_{j-1} \cdot \mathbf{y}_{j-1}}\mathbf{y}_{j-1} \\ &= \mathbf{x}_j - \mathbf{Proj}_{V_{j-1}}(\mathbf{x}_j). \end{aligned}$$

The Near Point Theorem

Developed here is the characterization of the orthogonal projection of a vector $\mathbf{x}$ onto a subspace V as the unique point $\mathbf{v}$ in V which minimizes $\|\mathbf{x} - \mathbf{v}\|$, that is, the point in V closest to $\mathbf{x}$.

In remembering the Gram-Schmidt formulas, and in the use of the orthogonal projection in proofs and constructions, the following key theorem is useful.

Theorem 18 (Orthogonal Projection Properties)
Let V be the span of orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

(a) Every vector in V has an orthogonal expansion $\mathbf{v} = \sum_{j=1}^{k} (\mathbf{v} \cdot \mathbf{u}_j)\mathbf{u}_j$.

(b) The vector $\mathbf{Proj}_V(\mathbf{x})$ is a vector in the subspace V.

(c) The vector $\mathbf{w} = \mathbf{x} - \mathbf{Proj}_V(\mathbf{x})$ is orthogonal to every vector in V.

(d) Among all vectors $\mathbf{v}$ in V, the minimum value of $\|\mathbf{x} - \mathbf{v}\|$ is uniquely obtained by the orthogonal projection $\mathbf{v} = \mathbf{Proj}_V(\mathbf{x})$.

Proof:
(a): Every element $\mathbf{v}$ in V is a linear combination of basis elements:

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k.$$

Take the dot product of this relation with basis element $\mathbf{u}_j$. By orthogonality, $c_j = \mathbf{v} \cdot \mathbf{u}_j$.

(b): Because $\mathbf{Proj}_V(\mathbf{x})$ is a linear combination of basis elements of V, then (b) holds.

(c): Let's compute the dot product of $\mathbf{w}$ and $\mathbf{v}$. We will use the orthogonal expansion from (a).

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{v} &= (\mathbf{x} - \mathbf{Proj}_V(\mathbf{x})) \cdot \mathbf{v} \\
&= \mathbf{x} \cdot \mathbf{v} - \left(\sum_{j=1}^{k} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \right) \cdot \mathbf{v} \\
&= \sum_{j=1}^{k} (\mathbf{v} \cdot \mathbf{u}_j)(\mathbf{u}_j \cdot \mathbf{x}) - \sum_{j=1}^{k} (\mathbf{x} \cdot \mathbf{u}_j)(\mathbf{u}_j \cdot \mathbf{v}) \\
&= 0.
\end{aligned}$$

(d): Begin with the Pythagorean identity

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2$$

valid exactly when $\mathbf{a} \cdot \mathbf{b} = 0$ (a right triangle, $\theta = 90°$). Using an arbitrary $\mathbf{v}$ in V, define $\mathbf{a} = \mathbf{Proj}_V(\mathbf{x}) - \mathbf{v}$ and $\mathbf{b} = \mathbf{x} - \mathbf{Proj}_V(\mathbf{x})$. By (b), vector $\mathbf{a}$ is in V. Because of (c), then $\mathbf{a} \cdot \mathbf{b} = 0$. This gives the identity

$$\|\mathbf{Proj}_V(\mathbf{x}) - \mathbf{v}\|^2 + \|\mathbf{x} - \mathbf{Proj}_V(\mathbf{x})\|^2 = \|\mathbf{x} - \mathbf{v}\|^2,$$

which establishes $\|\mathbf{x} - \mathbf{Proj}_V(\mathbf{x})\| < \|\mathbf{x} - \mathbf{v}\|$ except for the unique $\mathbf{v}$ such that $\|\mathbf{Proj}_V(\mathbf{x}) - \mathbf{v}\| = 0$.

The proof is complete.

Theorem 19 (Near Point to a Subspace)
Let V be a subspace of $\mathcal{R}^n$ and $\mathbf{x}$ a vector not in V. The **near point** to $\mathbf{x}$ in V is the orthogonal projection of $\mathbf{x}$ onto V. This point is characterized as the minimum of $\|\mathbf{x} - \mathbf{v}\|$ over all vectors $\mathbf{v}$ in the subspace V.

Proof: Apply **(d)** of the preceding theorem.

Theorem 20 (Cross Product and Projections)
The cross product direction $\mathbf{a} \times \mathbf{b}$ can be computed as $\mathbf{c} - \mathbf{Proj}_V(\mathbf{c})$, by selecting a direction $\mathbf{c}$ not in $V = \mathbf{span}\{\mathbf{a}, \mathbf{b}\}$.

Proof: The cross product makes sense only in $\mathcal{R}^3$. Subspace V is two dimensional when $\mathbf{a}$, $\mathbf{b}$ are independent, and Gram-Schmidt applies to find an orthonormal basis $\mathbf{u}_1$, $\mathbf{u}_2$. By (c) of Theorem 18, the vector $\mathbf{c} - \mathbf{Proj}_V(\mathbf{c})$ has the same or opposite direction to the cross product.

The QR Decomposition

The Gram-Schmidt formulas can be organized as matrix multiplication $A = QR$, where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the independent columns of A, and Q has columns equal to the Gram-Schmidt orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, which are the unitized Gram-Schmidt vectors.

Theorem 21 (The QR-Decomposition)
Let the $m \times n$ matrix A have independent columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then there is an upper triangular matrix R with positive diagonal entries and an orthonormal matrix Q such that

$$A = QR.$$

Proof: Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be the Gram-Schmidt orthogonal vectors given by relations on page 542. Define $\mathbf{u}_k = \mathbf{y}_k/\|\mathbf{y}_k\|$ and $r_{kk} = \|\mathbf{y}_k\|$ for $k = 1, \ldots, n$, and otherwise $r_{ij} = \mathbf{u}_i \cdot \mathbf{x}_j$. Let $Q = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$. Then

$$\begin{array}{rcl} \mathbf{x}_1 &=& r_{11}\mathbf{u}_1, \\ \mathbf{x}_2 &=& r_{22}\mathbf{u}_2 + r_{21}\mathbf{u}_1, \\ \mathbf{x}_3 &=& r_{33}\mathbf{u}_3 + r_{31}\mathbf{u}_1 + r_{32}\mathbf{u}_2, \\ &\vdots& \\ \mathbf{x}_n &=& r_{nn}\mathbf{u}_n + r_{n1}\mathbf{u}_1 + \cdots + r_{nn-1}\mathbf{u}_{n-1}. \end{array} \tag{2}$$

It follows from (2) and matrix multiplication that $A = QR$. The proof is complete.

Theorem 22 (Matrices Q and R in $A = QR$)
Let the $m \times n$ matrix A have independent columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be the Gram-Schmidt orthogonal vectors given by relations on page 542. Define $\mathbf{u}_k = \mathbf{y}_k/\|\mathbf{y}_k\|$. Then $AQ = QR$ is satisfied by $Q = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and

$$R = \begin{pmatrix} \|y_1\| & \mathbf{u}_1 \cdot \mathbf{x}_2 & \mathbf{u}_1 \cdot \mathbf{x}_3 & \cdots & \mathbf{u}_1 \cdot \mathbf{x}_n \\ 0 & \|y_2\| & \mathbf{u}_2 \cdot \mathbf{x}_3 & \cdots & \mathbf{u}_2 \cdot \mathbf{x}_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \|y_n\| \end{pmatrix}.$$

Proof: The result is contained in the proof of the previous theorem.
Some references cite the diagonal entries as $\|\mathbf{x}_1\|, \|\mathbf{x}_2^\perp\|, \ldots, \|\mathbf{x}_n^\perp\|$, where $\mathbf{x}_j^\perp = \mathbf{x}_j - \mathbf{Proj}_{V_{j-1}}(\mathbf{x}_j)$, $V_{j-1} = \mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\}$. Because $\mathbf{y}_1 = \mathbf{x}_1$ and $\mathbf{y}_j = \mathbf{x}_j - \mathbf{Proj}_{V_{j-1}}(\mathbf{x}_j)$, the formulas for R are identical.

Theorem 23 (Uniqueness of Q and R)
Let $m \times n$ matrix A have independent columns and satisfy the decomposition $A = QR$. If Q is $m \times n$ orthogonal and R is $n \times n$ upper triangular with positive diagonal elements, then Q and R are uniquely determined.

Proof: The problem is to show that $A = Q_1R_1 = Q_2R_2$ implies $R_2R_1^{-1} = I$ and $Q_1 = Q_2$. We start with $Q_1 = Q_2R_2R_1^{-1}$. Define $P = R_2R_1^{-1}$. Then $Q_1 = Q_2P$. Because $I = Q_1^TQ_1 = P^TQ_2^TQ_2P = P^TP$, then P is orthogonal. Matrix P is the product of square upper triangular matrices with positive diagonal elements, which implies P itself is square upper triangular with positive diagonal elements. The only matrix with these properties is the identity matrix I. Then $R_2R_1^{-1} = P = I$, which implies $R_1 = R_2$ and $Q_1 = Q_2$. The proof is complete.

Theorem 24 (Orthonormal Diagonal Form)
Let A be a given $n \times n$ real matrix. Then $A = QDQ^{-1}$ with Q orthogonal and D diagonal if and only if $A^T = A$.

Proof: The reader is reminded that Q orthogonal means that the columns of Q are **orthonormal**. The equation $A = A^T$ means A is **symmetric**.

Assume first that $A = QDQ^{-1}$ with $Q = Q^T$ orthogonal ($Q^TQ = I$) and D diagonal. Then $Q^T = Q = Q^{-1}$. This implies $A^T = (QDQ^{-1})^T = (Q^{-1})^TD^TQ^T = QDQ^{-1} = A$.

Conversely, assume $A^T = A$. Then the eigenvalues of A are real and eigenvectors corresponding to distinct eigenvalues are orthogonal. The proof proceeds by induction on the dimension n of the $n \times n$ matrix A.

For $n = 1$, let Q be the 1×1 identity matrix. Then Q is orthogonal and $AQ = QD$ where D is 1×1 diagonal.

Assume the decomposition $AQ = QD$ for dimension n. Let's prove it for A of dimension $n + 1$. Choose a real eigenvalue λ of A and eigenvector $\mathbf{v}_1$ with $\|\mathbf{v}_1\| = 1$. Complete a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ of $\mathcal{R}^{n+1}$. By Gram-Schmidt, we assume as well that this basis is orthonormal. Define $P = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$. Then P is orthogonal and satisfies $P^T = P^{-1}$. Define $B = P^{-1}AP$. Then B is symmetric ($B^T = B$) and $\mathbf{col}(B, 1) = \lambda\,\mathbf{col}(I, 1)$. These facts imply that B is a block matrix

$$B = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & C \end{array}\right)$$

where C is symmetric ($C^T = C$). The induction hypothesis applies to C to obtain the existence of an orthogonal matrix Q_1 such that $CQ_1 = Q_1D_1$ for some diagonal matrix D_1. Define a diagonal matrix D and matrices W and Q as follows:

$$\begin{aligned} D &= \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D_1 \end{array}\right), \\ W &= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_1 \end{array}\right), \\ Q &= PW. \end{aligned}$$

Then Q is the product of two orthogonal matrices, which makes Q orthogonal. Compute

$$W^{-1}BW = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_1^{-1} \end{array}\right)\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & C \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_1 \end{array}\right) = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D_1 \end{array}\right).$$

Then $Q^{-1}AQ = W^{-1}P^{-1}APW = W^{-1}BW = D$. This completes the induction, ending the proof of the theorem.

Theorem 25 (Eigenpairs of a Symmetric A)
Let A be a symmetric $n \times n$ real matrix. Then A has n eigenpairs $(\lambda_1, \mathbf{v}_1)$, $\ldots$, $(\lambda_n, \mathbf{v}_n)$, with independent eigenvectors $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_n$.

Proof: The preceding theorem applies to prove the existence of an orthogonal matrix Q and a diagonal matrix D such that $AQ = QD$. The diagonal entries of D are the eigenvalues of A, in some order. For a diagonal entry λ of D appearing in row j, the relation $A\,\mathbf{col}(Q, j) = \lambda\,\mathbf{col}(Q, j)$ holds, which implies that A has n eigenpairs. The eigenvectors are the columns of Q, which are orthogonal and hence independent. The proof is complete.

Theorem 26 (Diagonalization of Symmetric A)
Let A be a symmetric $n \times n$ real matrix. Then A has n eigenpairs. For each distinct eigenvalue λ, replace the eigenvectors by orthonormal eigenvectors, using the Gram-Schmidt process. Let $\mathbf{u}_1$, $\ldots$, $\mathbf{u}_n$ be the orthonormal vectors so obtained and define

$$Q = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_n), \quad D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n).$$

Then Q is orthogonal and $AQ = QD$.

Proof: The preceding theorem justifies the eigenanalysis result. Already, eigenpairs corresponding to distinct eigenvalues are orthogonal. Within the set of eigenpairs with the same eigenvalue λ, the Gram-Schmidt process produces a replacement basis of orthonormal eigenvectors. Then the union of all the eigenvectors is orthonormal. The process described here does not disturb the ordering of eigenpairs, because it only replaces an eigenvector. The proof is complete.

The Singular Value Decomposition

The decomposition has been used as a data compression algorithm. A geometric interpretation will be given in the next subsection.

Theorem 27 (Positive Eigenvalues of A^TA)
Given an $m \times n$ real matrix A, then A^TA is a real symmetric matrix whose eigenpairs $(\lambda, \mathbf{v})$ satisfy

$$\lambda = \frac{\|A\mathbf{v}\|^2}{\|\mathbf{v}\|^2} \geq 0. \tag{3}$$

Proof: Symmetry follows from $(A^TA)^T = A^T(A^T)^T = A^TA$. An eigenpair $(\lambda, \mathbf{v})$ satisfies $\lambda\overline{\mathbf{v}}^T\mathbf{v} = \overline{\mathbf{v}}^TA^TA\mathbf{v} = (\overline{A\mathbf{v}})^T(A\mathbf{v}) = \|A\mathbf{v}\|^2$, hence (3).

Definition 4 (Singular Values of A)
Let the real symmetric matrix A^TA have real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$. The numbers $\sigma_k = \sqrt{\lambda_k}$ $(1 \leq k \leq n)$ are called the **singular values** of the matrix A. The ordering of the singular values is always with decreasing magnitude.

Theorem 28 (Orthonormal Set $\mathbf{u}_1, \ldots, \mathbf{u}_m$)
Let the real symmetric matrix A^TA have real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$, obtained by the Gram-Schmidt process. Define the vectors

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1, \ldots, \mathbf{u}_r = \frac{1}{\sigma_r} A\mathbf{v}_r.$$

Because $\|A\mathbf{v}_k\| = \sigma_k$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is orthonormal. Gram-Schmidt can extend this set to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of $\mathcal{R}^m$.

Theorem 29 (The Singular Value Decomposition (svd))
Let A be a given real $m \times n$ matrix. Let $(\lambda_1, \mathbf{v}_1),\ldots,(\lambda_n, \mathbf{v}_n)$ be a set of orthonormal eigenpairs for A^TA such that $\sigma_k = \sqrt{\lambda_k}$ $(1 \leq k \leq r)$ defines the positive singular values of A and $\lambda_k = 0$ for $r < k \leq n$. Complete $\mathbf{u}_1 = (1/\sigma_1)A\mathbf{v}_1$, ..., $\mathbf{u}_r = (1/\sigma_r)A\mathbf{v}_r$ to an orthonormal basis $\{\mathbf{u}_k\}_{k=1}^m$ for $\mathcal{R}^m$. Define

$$U = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_m), \quad \Sigma = \left(\begin{array}{c|c} \mathbf{diag}(\sigma_1, \ldots, \sigma_r) & 0 \\ \hline 0 & 0 \end{array}\right),$$
$$V = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Then the columns of U and V are orthonormal and

$$\begin{aligned} A &= U\Sigma V^T \\ &= \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T \\ &= A(\mathbf{v}_1)\mathbf{v}_1^T + \cdots + A(\mathbf{v}_r)\mathbf{v}_r^T \end{aligned}$$

Proof of Theorem 28: Because $A^TA\mathbf{v}_k = \lambda_k\mathbf{v}_k \neq \mathbf{0}$ for $1 \leq k \leq r$, the vectors $\mathbf{u}_k$ are nonzero. Given $i \neq j$, then $\sigma_i\sigma_j\mathbf{u}_i \cdot \mathbf{u}_j = (A\mathbf{v}_i)^T(A\mathbf{v}_j) = \lambda_j\mathbf{v}_i^T\mathbf{v}_j = 0$, showing that the vectors $\mathbf{u}_k$ are orthogonal. Further, $\|\mathbf{u}_k\|^2 = \mathbf{v}_k \cdot (A^TA\mathbf{v}_k)/\lambda_k = \|\mathbf{v}_k\|^2 = 1$ because $\{\mathbf{v}_k\}_{k=1}^n$ is an orthonormal set.

The extension of the $\mathbf{u}_k$ to an orthonormal basis of $\mathcal{R}^m$ is not unique, because it depends upon a choice of independent spanning vectors $\mathbf{y}_{r+1}$, ..., $\mathbf{y}_m$ for the set $\{\mathbf{x} : \mathbf{x} \cdot \mathbf{u}_k = 0, \quad 1 \leq k \leq r\}$. Once selected, Gram-Schmidt is applied to $\mathbf{u}_1$, ..., $\mathbf{u}_r$, $\mathbf{y}_{r+1}$, ..., $\mathbf{y}_m$ to obtain the desired orthonormal basis.

Proof of Theorem 29: The product of U and Σ is the $m \times n$ matrix

$$\begin{aligned} U\Sigma &= \mathbf{aug}(\sigma_1\mathbf{u}_1, \ldots, \sigma_r\mathbf{u}_r, \mathbf{0}, \ldots, \mathbf{0}) \\ &= \mathbf{aug}(A(\mathbf{v}_1), \ldots, A(\mathbf{v}_r), \mathbf{0}, \ldots, \mathbf{0}). \end{aligned}$$

Let $\mathbf{v}$ be any vector in $\mathcal{R}^n$. It will be shown that $U\Sigma V^T\mathbf{v}$, $\sum_{k=1}^r A(\mathbf{v}_k)(\mathbf{v}_k^T\mathbf{v})$ and $A\mathbf{v}$ are the same column vector. We have the equalities

$$\begin{aligned} U\Sigma V^T\mathbf{v} &= U\Sigma \begin{pmatrix} \mathbf{v}_1^T\mathbf{v} \\ \vdots \\ \mathbf{v}_n^T\mathbf{v} \end{pmatrix} \\ &= \mathbf{aug}(A(\mathbf{v}_1), \ldots, A(\mathbf{v}_r), \mathbf{0}, \ldots, \mathbf{0}) \begin{pmatrix} \mathbf{v}_1^T\mathbf{v} \\ \vdots \\ \mathbf{v}_n^T\mathbf{v} \end{pmatrix} \\ &= \sum_{k=1}^r (\mathbf{v}_k^T\mathbf{v})A(\mathbf{v}_k). \end{aligned}$$

Because $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthonormal basis of $\mathcal{R}^n$, then $\mathbf{v} = \sum_{k=1}^{n}(\mathbf{v}_k^T\mathbf{v})\mathbf{v}_k$. Additionally, $A(\mathbf{v}_k) = \mathbf{0}$ for $r < k \leq n$ implies

$$\begin{aligned} A\mathbf{v} &= A\left(\sum_{k=1}^{n}(\mathbf{v}_k^T\mathbf{v})\mathbf{v}_k\right) \\ &= \sum_{k=1}^{r}(\mathbf{v}_k^T\mathbf{v})A(\mathbf{v}_k) \end{aligned}$$

Then $A\mathbf{v} = U\Sigma V^T\mathbf{v} = \sum_{k=1}^{r} A(\mathbf{v}_k)(\mathbf{v}_k^T\mathbf{v})$, which proves the theorem.

Singular Values and Geometry

Discussed here is how to interpret singular values geometrically, especially in low dimensions 2 and 3. First, we review conics, adopting the viewpoint of eigenanalysis.

Standard equation of an ellipse. Calculus courses consider ellipse equations like $85x^2-60xy+40y^2 = 2500$ and discuss removal of the cross term $-60xy$. The objective is to obtain a **standard** ellipse equation $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$. We re-visit this old problem from a different point of view, and in the derivation establish a connection between the ellipse equation, the symmetric matrix A^TA, and the singular values of A.

9 Example (Image of the Unit Circle) Let $A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$. Then the invertible matrix A maps the unit circle K into the ellipse

$$85x^2 - 60xy + 40y^2 = 2500.$$

Verify that after a rotation to remove the xy-term, in the new XY-coordinates the equation is $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$, where $a = 10$ and $b = 5$.

Solution: The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ will be used together with the parameterization $\theta \to (\cos\theta, \sin\theta)$ of the unit circle K, $0 \leq \theta \leq 2\pi$. Mapping K by the matrix A is formally the set of dual relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The Pythagorean identity used on the second relation implies

$$85x^2 - 60xy + 40y^2 = 2500.$$

This ellipse equation can be represented by the vector-matrix identity

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 85 & 30 \\ 30 & 40 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2500.$$

The symmetric matrix $A^TA = \begin{pmatrix} 85 & 30 \\ 30 & 40 \end{pmatrix}$ has eigenpair packages

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix}.$$

In the coordinate system $\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$ of the orthogonal matrix P, the ellipse vector-matrix identity becomes

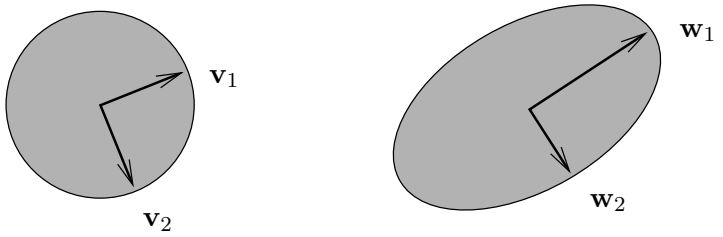
$$\begin{pmatrix} X & Y \end{pmatrix} P^T \begin{pmatrix} 85 & 30 \\ 30 & 40 \end{pmatrix} P \begin{pmatrix} X \\ Y \end{pmatrix} = 2500.$$

Because $P^T \left(A^TA\right) P = D = \textbf{diag}(25, 100)$, then the ellipse equation has the standard form

$$25X^2 + 100Y^2 = 2500.$$

The semi-axis lengths for this ellipse are $a = \sqrt{\frac{2500}{25}} = 10$ and $b = \sqrt{\frac{2500}{100}} = 5$, which are precisely the singular values $\sigma_1 = 10$ and $\sigma_2 = 5$ of matrix A.

Geometry. The preceding example is typical for all invertible 2×2 matrices A. Described here is the geometrical interpretation for the singular value decomposition $A = U\Sigma V^T$, shown in Figure 4.



Unit circle K Image $A(K)$

Figure 4. Mapping the unit circle.
Invertible matrix A maps the unit circle K into the ellipse $A(K)$. Orthonormal vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are mapped by matrix $A = U\Sigma V^T$ into orthogonal vectors $\mathbf{w}_1 = A\mathbf{v}_1$, $\mathbf{w}_2 = A\mathbf{v}_2$, which are the semi-axes vectors of the ellipse. The semi-axis lengths $\|\mathbf{w}_1\|$, $\|\mathbf{w}_2\|$ equal the singular values σ_1, σ_2.

A summary of the example $A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$:

> A 2×2 invertible matrix A maps the unit circle K into an ellipse $A(K)$. Decomposition $A = U\Sigma V^T$ means A maps the columns of V into re-scaled columns of U. These vectors, $\sigma_1\mathbf{u}_1$ and $\sigma_2\mathbf{u}_2$, are the semi-axis vectors of the ellipse $A(K)$, whose lengths σ_1, σ_2 are the singular values.

The columns of V are orthonormal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, computed as eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$ of A^TA. Then $A\mathbf{v}_1 = U\Sigma V^T\mathbf{v}_1 = U \begin{pmatrix} \sigma_1 \\ 0 \end{pmatrix} = \sigma_1\mathbf{u}_1$. Similarly, $A\mathbf{v}_2 = U\Sigma V^T\mathbf{v}_2 = U \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix} = \sigma_2\mathbf{u}_2$.