Linear Nonhomogeneous System

Given numbers a_{11}, ..., a_{mn}, b_1, ..., b_m, consider the **nonhomogeneous system** of m linear equations in n **unknowns** x_1, x_2, ..., x_n

$$(1)\quad \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2, \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m. \end{array}$$

Constants a_{11}, ..., a_{mn} are called the **coefficients** of system (1). Constants b_1, ..., b_m are collectively referenced as the **right hand side**, **right side** or **RHS**.

Linear Homogeneous System

Given numbers $a_{11}, \ldots, a_{mn}$ consider the **homogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$

$$(2)\quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (2).

Definition 2 (Parametric Equations)
The terminology **parametric equations** refers to a set of equations of the form

$$(3)\quad \begin{array}{rcl} x_1 &=& d_1 + c_{11}t_1 + \cdots + c_{1k}t_k, \\ x_2 &=& d_2 + c_{21}t_1 + \cdots + c_{2k}t_k, \\ & \vdots & \\ x_n &=& d_n + c_{n1}t_1 + \cdots + c_{nk}t_k. \end{array}$$

The numbers $d_1, \ldots, d_n, c_{11}, \ldots, c_{nk}$ are *known constants* and the variable names $t_1, \ldots, t_k$ are **parameters**. The symbols $t_1, \ldots, t_k$ are therefore allowed to take on any value from $-\infty$ to ∞.

Definition 3 (General Solution)
A **general solution** (sometimes called a **parametric solution**) of (1) is a set of parametric equations (3) plus two additional requirements:

(4) Equations (3) satisfy (1) for all real values of $t_1, \ldots, t_k$.

(5) Any solution of (1) can be obtained from (3) by specializing values of the parameters $t_1, t_2, \ldots t_k$.

Reduced Echelon Systems

A system of linear algebraic equations in which each nonzero equation has a **lead variable** is called a **reduced echelon system**. By convention, the equations with lead variables are listed in the variable list order. Following them are any zero equations.

A **lead variable** is a variable that appears first (left-to-right) with coefficient one in exactly one equation.

A **free variable** in a reduced echelon system is any variable that is not a lead variable.

Recognition of Reduced Echelon Systems

A reduced echelon system has the special form

$$
(6) \quad \begin{array}{ccccccccc}
x_{i_1} & + & E_{11}x_{j_1} & + & \cdots & + & E_{1k}x_{j_k} & = & D_1, \\
x_{i_2} & + & E_{21}x_{j_1} & + & \cdots & + & E_{2k}x_{j_k} & = & D_2, \\
 & & & & & & & \vdots & \\
x_{i_m} & + & E_{m1}x_{j_1} & + & \cdots & + & E_{mk}x_{j_k} & = & D_m.
\end{array}
$$

The numbers $E_{11}, \ldots, E_{mk}$ and $D_1, \ldots, D_m$ are *known constants*.

A linear system (1) is recognized as a reduced echelon system exactly when the first variable listed in each equation has coefficient one and that variable name appears nowhere else in the system.

Writing a Standard Parametric Solution

Consider the **reduced echelon system**

$$(7)\quad \begin{array}{rcl} \boxed{x} + 4w + u + v & = & 1, \\ \boxed{y} - u + v & = & 2, \\ \boxed{z} - w + 2u - v & = & 0. \end{array}$$

To write out the parametric solution,

1. Set the free variables equal to invented *parameter names* $t_1, \ldots, t_k$, where $-\infty < t_j < \infty$, $1 \le j \le k$.

2. Solve equations for the leading variables and back-substitute the free variables to obtain a **standard parametric solution**.

Writing a Standard Parametric Solution

$$
(8)\quad
\begin{array}{rcl}
\boxed{x} + 4w + u + v & = & 1, \\
\boxed{y} - u + v & = & 2, \\
\boxed{z} - w + 2u - v & = & 0.
\end{array}
$$

The boxed **lead variables** in (8) are x, y, z and the **free variables** are w, u, v. Assign parameters t_1, t_2, t_3 to the free variables and back-substitute in (8) to obtain a **standard parametric solution**

$$
\begin{cases}
x = 1 - 4t_1 - t_2 - t_3, \\
y = 2 + t_2 - t_3 \\
z = t_1 - 2t_2 + t_3, \\
w = t_1, \\
u = t_2, \\
v = t_3.
\end{cases}
$$

Writing a Standard Parametric Solution

By convention, the general solution lists the variables in list order x, w, u, v, y, z.

$$\begin{cases} x &= 1 - 4t_1 - t_2 - t_3, \\ w &= t_1, \\ u &= t_2, \\ v &= t_3, \\ y &= 2 + t_2 - t_3, \\ z &= t_1 - 2t_2 + t_3. \end{cases}$$

Three Rules for Equivalent Systems

The following rules *neither create nor destroy solutions* of the original system.

- **Swap** Two equations can be interchanged without changing the solution set.
- **Mult** An equation can be multiplied by $c \neq 0$ without changing the solution set.
- **Combo** A multiple of one equation can be added to a different equation without changing the solution set.

The last two rules replace an existing equation by a new one. The **mult** rule is reversed by multiplication by $1/c$, whereas the **combo** rule is reversed by subtracting the equation–multiple previously added. In short, the three operations are **reversible**.

Gaussian Elimination

This algorithm applies at each algebraic step one of the **three rules** defined above in : **mult**, **swap** and **combo**.

The objective of each algebraic step is to **increase the number of lead variables**. The process stops when no more lead variables can be found, in which case the last system of equations is a **reduced echelon system**.

Reversibility of the algebraic steps means that no solutions are created or destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have *exactly the same solutions*.

The final reduced echelon system has an easily–found standard parametric solution, which is reported as the **general solution**.

Theorem 3 (Gaussian Elimination)
Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the three rules of **swap**, **mult** and **combo**.