Systems of Differential Equations
Matrix Methods

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Characteristic Equation

Definition 1 (Characteristic Equation)
Given a square matrix A, the **characteristic equation** of A is the polynomial equation

$$\det(A - rI) = 0.$$

The determinant $\det(A - rI)$ is formed by subtracting r from the diagonal of A.
The polynomial $p(r) = \det(A - rI)$ is called the **characteristic polynomial**.

- If A is 2×2, then $p(r)$ is a quadratic.
- If A is 3×3, then $p(r)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

Characteristic Equation Examples

Create $\det(\boldsymbol{A} - r\boldsymbol{I})$ by subtracting $\boldsymbol{r}$ from the diagonal of $\boldsymbol{A}$.

Evaluate by the cofactor rule.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2-r & 3 \\ 0 & 4-r \end{vmatrix} = (2-r)(4-r)$$

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2-r & 3 & 4 \\ 0 & 5-r & 6 \\ 0 & 0 & 7-r \end{vmatrix} = (2-r)(5-r)(7-r)$$

Cayley-Hamilton

Theorem 1 (Cayley-Hamilton)
A square matrix A satisfies its own characteristic equation.

If $p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots a_0$, then the result is the equation

$$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,$$

where I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix.

The 2×2 Case

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \mathbf{trace}(A)$, $a_0 = \det(A)$ we have $p(r) = r^2 + a_1(-r) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Cayley-Hamilton Example

Assume

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}$$

Then

$$\boldsymbol{p}(\boldsymbol{r}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{r} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} - \boldsymbol{r} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} - \boldsymbol{r} \end{vmatrix} = (\mathbf{2} - \boldsymbol{r})(\mathbf{5} - \boldsymbol{r})(\mathbf{7} - \boldsymbol{r})$$

and the Cayley-Hamilton Theorem says that

$$(\mathbf{2}\boldsymbol{I} - \boldsymbol{A})(\mathbf{5}\boldsymbol{I} - \boldsymbol{A})(\mathbf{7}\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Cayley-Hamilton-Ziebur Theorem

Theorem 2 (Cayley-Hamilton-Ziebur Structure Theorem for $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$)
A component function $u_k(t)$ of the vector solution $\vec{\mathbf{u}}(t)$ for $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$ is a solution of the nth order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\det(A - rI) = 0$.

Meaning: The vector solution $\vec{\mathbf{u}}(t)$ of

$$\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$$

is a vector linear combination of the Euler solution atoms constructed from the roots of the characteristic equation $\det(A - rI) = 0$.

Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case $n = 2$, because the proof details are similar in higher dimensions.

$r^2 + a_1 r + a_0 = 0$	Expanded characteristic equation
$A^2 + a_1 A + a_0 I = 0$	Cayley-Hamilton matrix equation
$A^2\vec{u} + a_1 A\vec{u} + a_0\vec{u} = \vec{0}$	Right-multiply by $\vec{u} = \vec{u}(t)$
$\vec{u}'' = A\vec{u}' = A^2\vec{u}$	Differentiate $\vec{u}' = A\vec{u}$
$\vec{u}'' + a_1\vec{u}' + a_0\vec{u} = \vec{0}$	Replace $A^2\vec{u} \to \vec{u}''$, $A\vec{u} \to \vec{u}'$

Then the components $x(t)$, $y(t)$ of $\vec{u}(t)$ satisfy the two differential equations

$$\begin{aligned} x''(t) + a_1 x'(t) + a_0 x(t) &= 0, \\ y''(t) + a_1 y'(t) + a_0 y(t) &= 0. \end{aligned}$$

This system implies that the components of $\vec{u}(t)$ are solutions of the second order DE with characteristic equation $\det(A - rI) = 0$.

The Cayley-Hamilton-Ziebur Method for $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

Let $\mathbf{atom}_1, \ldots, \mathbf{atom}_n$ denote the Euler solution atoms constructed from the nth order characteristic equation $\det(A - rI) = 0$ by Euler's Theorem. The solution of

$$\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$$

is given for some constant vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ by the equation

$$\vec{\mathbf{u}}(t) = (\mathbf{atom}_1)\vec{\mathbf{d}}_1 + \cdots + (\mathbf{atom}_n)\vec{\mathbf{d}}_n$$

Warning: The vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ are not arbitrary; they depend on the n initial conditions $u_k(0) = c_k, k = 1, \ldots, n$.

Cayley-Hamilton-Ziebur Method Conclusions

- Solving $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is reduced to finding the constant vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$.
- The vectors $\vec{\mathbf{d}}_j$ are **not arbitrary**. They are **uniquely determined** by A and $\vec{\mathbf{u}}(0)$!

  A general method to find them is to differentiate the equation

  $$\vec{\mathbf{u}}(t) = (\mathbf{atom}_1)\vec{\mathbf{d}}_1 + \cdots + (\mathbf{atom}_n)\vec{\mathbf{d}}_n$$

  $n-1$ times, then set $t = 0$ and replace $\vec{\mathbf{u}}^{(k)}(0)$ by $A^k\mathbf{u}(0)$ [because $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}' = AA\vec{\mathbf{u}}$, etc]. The resulting n equations in vector unknowns $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ can be solved by elimination.
- If all atoms constructed are base atoms constructed from real roots, then each $\vec{\mathbf{d}}_j$ is a constant multiple of a real eigenvector of A. Atom e^{rt} corresponds to the eigenpair equation $A\mathbf{v} = r\mathbf{v}$.

A 2×2 Illustration

Let's solve $\vec{\mathbf{u}}' = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{pmatrix} \vec{\mathbf{u}}$, $\quad \mathbf{u}(0) = \begin{pmatrix} -\mathbf{1} \\ \mathbf{2} \end{pmatrix}$.

The characteristic polynomial of the non-triangular matrix $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{pmatrix}$ is

$$\begin{vmatrix} \mathbf{1} - \boldsymbol{r} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} - \boldsymbol{r} \end{vmatrix} = (\mathbf{1} - \boldsymbol{r})^2 - \mathbf{4} = (\boldsymbol{r} + \mathbf{1})(\boldsymbol{r} - \mathbf{3}).$$

Euler's theorem implies solution atoms are $\boldsymbol{e}^{-t}$, $\boldsymbol{e}^{3t}$.

Then $\vec{\mathbf{u}}$ is a vector linear combination of the solution atoms,

$$\vec{\mathbf{u}} = \boldsymbol{e}^{-t}\vec{\mathbf{d}}_1 + \boldsymbol{e}^{3t}\vec{\mathbf{d}}_2.$$

How to Find $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$

We solve for vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ in the equation

$$\vec{\mathbf{u}} = e^{-t}\vec{\mathbf{d}}_1 + e^{3t}\vec{\mathbf{d}}_2.$$

Advice: Define$\vec{\mathbf{d}}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Differentiate the above relation. Replace $\vec{\mathbf{u}}'$ via $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, then set $t = 0$ and replace $\vec{\mathbf{u}}(0)$ by $\vec{\mathbf{d}}_0$ in the two formulas to obtain the relations

$$\begin{aligned} \vec{\mathbf{d}}_0 &= e^0\vec{\mathbf{d}}_1 + e^0\vec{\mathbf{d}}_2 \\ A\vec{\mathbf{d}}_0 &= -e^0\vec{\mathbf{d}}_1 + 3e^0\vec{\mathbf{d}}_2 \end{aligned}$$

We solve for $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ by elimination. Adding the equations gives $\vec{\mathbf{d}}_0 + A\vec{\mathbf{d}}_0 = 4\vec{\mathbf{d}}_2$ and then $\vec{\mathbf{d}}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ implies

$$\begin{aligned} \vec{\mathbf{d}}_1 &= \tfrac{3}{4}\vec{\mathbf{d}}_0 - \tfrac{1}{4}A\vec{\mathbf{d}}_0 = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}, \\ \vec{\mathbf{d}}_2 &= \tfrac{1}{4}\vec{\mathbf{d}}_0 + \tfrac{1}{4}A\vec{\mathbf{d}}_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \end{aligned}$$

Summary of the 2×2 Illustration

The solution of the dynamical system

$$\vec{\mathbf{u}}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{\mathbf{u}}, \quad \mathbf{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

is a vector linear combination of solution atoms e^{-t}, e^{3t} given by the equation

$$\vec{\mathbf{u}} = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Eigenpairs for Free

Each vector appearing in the formula is a scalar multiple of an eigenvector, because eigenvalues $-1, 3$ are real and distinct. The simplified eigenpairs are

$$\left(-1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

A Matrix Method for Finding $\vec{\mathrm{d}}_1$ and $\vec{\mathrm{d}}_2$

The Cayley-Hamilton-Ziebur Method produces a unique solution for $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ because the coefficient matrix

$$\begin{pmatrix} e^0 & e^0 \\ -e^0 & 3e^0 \end{pmatrix}$$

is exactly the Wronskian W of the basis of atoms e^{-t}, e^{3t} evaluated at $t = 0$. This same fact applies no matter the number of coefficients $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \ldots$ to be determined.

Let $\vec{d}_0 = \vec{u}(0)$, the initial condition. The answer for $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$ can be written in matrix form in terms of the transpose W^T of the Wronskian matrix as

$$\langle \vec{\mathrm{d}}_1 | \vec{\mathrm{d}}_2 \rangle = \langle \vec{\mathrm{d}}_0 | A\vec{\mathrm{d}}_0 \rangle (W^T)^{-1}.$$

Symbol $\langle \vec{A} | \vec{B} \rangle$ is the augmented matrix of column vecotrs $\vec{A}, \vec{B}$.

Solving a 2 × 2 Initial Value Problem by the Matrix Method

$$\vec{\mathbf{u}}' = A\vec{\mathbf{u}}, \quad \vec{\mathbf{u}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\vec{\mathbf{d}}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $A\vec{\mathbf{d}}_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and

$$\langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 \rangle = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^T \right)^{-1} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

Extract $\vec{d}_1 = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$, $\vec{d}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Then the solution of the initial value problem is

$$\vec{\mathbf{u}}(t) = e^{-t}\begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix} + e^{3t}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \\ \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}.$$

Other Representations of the Solution $\vec{\mathbf{u}}$

Let $y_1(t), \ldots, y_n(t)$ be a solution basis for the nth order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\det(A - rI) = 0$.

Consider the solution basis $\mathbf{atom}_1, \mathbf{atom}_2, \ldots, \mathbf{atom}_n$. Each atom is a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Replacing the atoms in the formula

$$\vec{\mathbf{u}}(t) = (\mathbf{atom}_1)\vec{\mathbf{d}}_1 + \cdots + (\mathbf{atom}_n)\vec{\mathbf{d}}_n$$

by these linear combinations implies there are constant vectors $\vec{\mathbf{D}}_1, \ldots, \vec{\mathbf{D}}_n$ such that

$$\vec{\mathbf{u}}(t) = y_1(t)\vec{\mathbf{D}}_1 + \cdots + y_n(t)\vec{\mathbf{D}}_n$$

Another General Solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

Theorem 3 (General Solution)
The unique solution of $\vec{\mathbf{u}}' = A\mathbf{u}$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{d}}_0$ is

$$\mathbf{u}(t) = \phi_1(t)\mathbf{u}_0 + \phi_2(t)A\mathbf{u}_0 + \cdots + \phi_n(t)A^{n-1}\mathbf{u}_0$$

where $\phi_1, \ldots, \phi_n$ are linear combinations of atoms constructed from roots of the characteristic equation $\det(A - rI) = 0$, such that

$$\mathbf{Wronskian}(\phi_1(t), \ldots, \phi_n(t))|_{t=0} = I.$$

Proof of the theorem

Proof: Details will be given for $n = 3$. The details for arbitrary matrix dimension n is a routine modification of this proof. The Wronskian condition implies ϕ_1, ϕ_2, ϕ_3 are independent. Then each atom constructed from the characteristic equation is a linear combination of ϕ_1, ϕ_2, ϕ_3. It follows that the unique solution $\vec{\mathbf{u}}$ can be written for some vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, $\vec{\mathbf{d}}_3$ as

$$\vec{\mathbf{u}}(t) = \phi_1(t)\vec{\mathbf{d}}_1 + \phi_2(t)\vec{\mathbf{d}}_2 + \phi_3(t)\vec{\mathbf{d}}_3.$$

Differentiate this equation twice and then set $t = 0$ in all 3 equations. The relations $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}' = AA\vec{\mathbf{u}}$ imply the 3 equations

$$\begin{array}{rcccccc}
\vec{\mathbf{d}}_0 & = & \phi_1(0)\vec{\mathbf{d}}_1 & + & \phi_2(0)\vec{\mathbf{d}}_2 & + & \phi_3(0)\vec{\mathbf{d}}_3 \\
A\vec{\mathbf{d}}_0 & = & \phi_1'(0)\vec{\mathbf{d}}_1 & + & \phi_2'(0)\vec{\mathbf{d}}_2 & + & \phi_3'(0)\vec{\mathbf{d}}_3 \\
A^2\vec{\mathbf{d}}_0 & = & \phi_1''(0)\vec{\mathbf{d}}_1 & + & \phi_2''(0)\vec{\mathbf{d}}_2 & + & \phi_3''(0)\vec{\mathbf{d}}_3
\end{array}$$

Because the Wronskian is the identity matrix I, then these equations reduce to

$$\begin{array}{rcccccc}
\vec{\mathbf{d}}_0 & = & 1\vec{\mathbf{d}}_1 & + & 0\vec{\mathbf{d}}_2 & + & 0\vec{\mathbf{d}}_3 \\
A\vec{\mathbf{d}}_0 & = & 0\vec{\mathbf{d}}_1 & + & 1\vec{\mathbf{d}}_2 & + & 0\vec{\mathbf{d}}_3 \\
A^2\vec{\mathbf{d}}_0 & = & 0\vec{\mathbf{d}}_1 & + & 0\vec{\mathbf{d}}_2 & + & 1\vec{\mathbf{d}}_3
\end{array}$$

which implies $\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_0$, $\vec{\mathbf{d}}_2 = A\vec{\mathbf{d}}_0$, $\vec{\mathbf{d}}_3 = A^2\vec{\mathbf{d}}_0$.
The claimed formula for $\vec{\mathbf{u}}(t)$ is established and the proof is complete.

Change of Basis Equation

Illustrated here is the change of basis formula for $n = 3$. The formula for general n is similar.

Let $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ denote the linear combinations of atoms obtained from the vector formula

$$\big(\phi_1(t), \phi_2(t), \phi_3(t)\big) = \big(\mathbf{atom}_1(t), \mathbf{atom}_2(t), \mathbf{atom}_3(t)\big)\, C^{-1}$$

where

$$C = \mathbf{Wronskian}(\mathbf{atom}_1, \mathbf{atom}_2, \mathbf{atom}_3)(0).$$

The solutions $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ are called the **principal solutions** of the linear homogeneous constant-coefficient differential equation constructed from the characteristic equation $\det(A - rI) = 0$. They satisfy the initial conditions

$$\mathbf{Wronskian}(\phi_1, \phi_2, \phi_3)(0) = I.$$