Undetermined Coefficients, Resonance, Applications

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An Undetermined Coefficients Illustration

The differential equation $y'' - y = x + xe^x$ will be solved. Verified for general solution $y = y_h + y_p$ are the formulas

$$y_h = c_1 e^x + c_2 e^{-x}, \quad y_p = -x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.$$

Homogeneous solution. The homogeneous equation $y'' - y = 0$ has characteristic equation $r^2 - 1 = 0$, roots $r = \pm 1$, and atom list e^x, e^{-x}. Then $y_h = c_1 e^x + c_2 e^{-x}$.

Rule I trial solution. Let $f(x) = x + xe^x$. The derivatives f, f', f'', …have atom list

$$1, \quad x, \quad e^x, \quad xe^x.$$

Then $k = 4$ is the number of atoms in the trial solution. Because atom e^x is a solution of the homogeneous equation $y'' - y = 0$, then Rule I **FAILS**.

Rule II trial solution. Break up the atom list for f into groups with the same base atom, as follows.

Group	Atoms	Base atom
1	$1, x$	e^{0x}
2	e^x, xe^x	e^x

Group 1 is unchanged, because the first atom 1 is not a solution of the homogeneous equation $y'' - y = 0$. Group 2 has a FAIL, because the first atom e^x is a solution of the homogeneous equation $y'' - y = 0$, as seen in Rule I. We multiply group 2 by factor x, then the first atom is xe^x, which is not a solution of the homogeneous equation [Eulers theorem says xe^x is a solution if and only if 1 is a double root of the characteristic equation $r^2 - 1 = 0$; it isn't].

Group	Atoms
1	$1, x$
New 2	xe^x, x^2e^x

The trial solution, according to Rule II, is a linear combination of the atoms in the last table.

$$y = (d_1 + d_2 x) + (d_3 x e^x + d_4 x^2 e^x).$$

Substitute the trial solution into the DE

Substitute $y(d_1 + d_2x) + (d_3xe^x + d_4x^2e^x)$ **into** $y'' - y = x + xe^x$. The details:

$\text{LHS} = y'' - y$	Left side of the equation.
$= [y_1'' - y_1] + [y_2'' - y_2]$	Let $y = y_1 + y_2$, $y_1 = d_1 + d_2x$, $y_2 = d_3xe^x + d_4x^2e^x$.
$= [0 - y_1] + [2d_3e^x + 2d_4e^x + 4d_4xe^x]$	Use $y_1'' = 0$ and $y_2'' = y_2 + 2d_3e^x + 2d_4e^x + 4d_4xe^x$.
$= (-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x$	Collect on distinct atoms.

Write out a 4×4 system. Because LHS $=$ RHS and RHS $= \boldsymbol{x + xe^x}$, the last display gives the relation

$$\begin{aligned}(1)\qquad & \boldsymbol{(-d_1)1 + (-d_2)x +} \\ & \boldsymbol{(2d_3 + 2d_4)e^x + (4d_4)xe^x \;=\; (0)1 + (1)x + (0)e^x + (1)xe^x.}\end{aligned}$$

Equate coefficients of matching atoms left and right to give the system of equations

$$(2)\qquad \begin{array}{rrrrcl} \boldsymbol{-d_1} & & & & \boldsymbol{=} & \boldsymbol{0,} \\ & \boldsymbol{-d_2} & & & \boldsymbol{=} & \boldsymbol{1,} \\ & & \boldsymbol{2d_3} & \boldsymbol{+2d_4} & \boldsymbol{=} & \boldsymbol{0,} \\ & & & \boldsymbol{4d_4} & \boldsymbol{=} & \boldsymbol{1.} \end{array}$$

Atom matching effectively removes $\boldsymbol{x}$ and changes the equation into a $\boldsymbol{4 \times 4}$ linear system for symbols $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$, $\boldsymbol{d_4}$.

Atom Matching Explained. The technique is independence. To explain, independence of atoms means that a linear combination of atoms is uniquely represented, hence two such equal representations must have matching coefficients. Relation (1) says that two linear combinations of the same list of atoms are equal. Hence coefficients left and right in (1) must match, which gives 4×4 system (2).

Solve the equations. The 4×4 system must always have a unique solution. Equivalently, there are four lead variables and zero free variables. Solving by back-substitution gives $d_1 = 0, d_2 = -1, d_3 = 1/4, d_4 = -1/4$.

Report y_p. The trial solution with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Report $\boldsymbol{y = y_h + y_p}$

From above,

$$y_h = c_1 e^{x} + c_2 e^{-x}, \quad y_p = -x - \frac{1}{4} x e^{x} + \frac{1}{4} x^2 e^{x}.$$

Then $y = y_h + y_p$ is given by

$$y = c_1 e^{x} + c_2 e^{-x} - x - \frac{1}{4} x e^{x} + \frac{1}{4} x^2 e^{x}.$$

Answer check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```

Pure Resonance

Graphed in Figure 7 are the envelope curves $x = \pm t$ and the solution $x(t) = t\sin 4t$ of the equation $x''(t) + 16x(t) = 8\cos\omega t$, where $\omega = 4$.

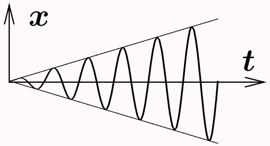

Figure 1. Pure resonance.

The notion of **pure resonance** in the differential equation

$$x''(t) + \omega_0^2\, x(t) = F_0 \cos(\omega t) \tag{3}$$

is the existence of a solution that is unbounded as $t \to \infty$. We already know that for $\omega \neq \omega_0$, the general solution of (6) is the sum of two harmonic oscillations, hence it is bounded. Equation (6) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution $x(t) = \dfrac{F_0}{2\omega_0} t\, \sin(\omega_0\, t)$. To summarize:

> Pure resonance occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω, in which case all solutions of the differential equation are unbounded.

In Figure 7, this is illustrated for $x''(t) + 16x(t) = 8\cos 4t$, which in (6) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$.

Real-World Damping Effects

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching

$$\omega = \omega_0 \equiv \sqrt{k/m}$$

characterizes the event. This ideal situation never happens in the physical world, because *damping is always present*. In the presence of damping $c > 0$, it can be established that *only bounded solutions exist* for the forced spring-mass system

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \tag{4}$$

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible solution amplitude. It can be shown that this happens for the condition

$$\omega = \sqrt{k/m - c^2/(2m^2)}, \quad k/m - c^2/(2m^2) > 0. \tag{5}$$

Pure resonance $\omega = \omega_0 \equiv \sqrt{k/m}$ is the limiting case obtained by setting the damping constant c to zero in condition (8). This strange but predictable interaction exists between the damping constant c and the size of solutions, relative to the external frequency ω, even though all solutions remain bounded.