The Corrected Trial Solution
in
the Method of Undetermined Coefficients

- Definition of Related Atoms
- The Basic Trial Solution Method
- Symbols
- Superposition
- Annihilator Polynomial for $f(x)$
- Annihilator Equation for $f(x)$
- Correction rule I: The Annihilator Method
- Correction rule II
- Correction rule III
- Definition of function atomRoot
- Correction rule IV

Definition of Related Atoms

Atoms $\boldsymbol{A}$ and $\boldsymbol{B}$ are *related* if and only if their successive derivatives share a common atom. Then $\boldsymbol{x^3}$ is related to $\boldsymbol{x}$ and $\boldsymbol{x^{101}}$, while $\boldsymbol{x}$ is unrelated to $\boldsymbol{e^x}$, $\boldsymbol{xe^x}$ and $\boldsymbol{x \sin x}$. Atoms $\boldsymbol{x \sin x}$ and $\boldsymbol{x^3 \cos x}$ are related, but the atoms $\boldsymbol{\cos 2x}$ and $\boldsymbol{\sin x}$ are unrelated.

The Basic Trial Solution Method

The method is outlined here for a second order differential equation $ay'' + by' + cy = f(x)$. The method applies unchanged for nth order equations.

Step **1**. Extract all distinct atoms from $f(x)$, $f'(x)$, $f''(x)$, ... to construct a maximal list of k atoms. Multiply these atoms by **undetermined coefficients** d_1, d_2, ..., d_k, then add, defining **trial solution** y.

Step **2**. Substitute y into the differential equation.

Basic Correction Rule. If some variable d_p is missing in the substituted equation, then step 2 fails. Correct the trial solution as follows. Variable d_p appears in trial solution y as term d_pA, where A is an atom. Multiply A and all its related atoms B by x. The modified expression y is called a **corrected trial solution**. Repeat step 2 until the substituted equation contains all of the variables $d_1, \ldots, d_k$.

Step **3**. Match coefficients of atoms left and right to write out linear algebraic equations for d_1, d_2, ..., d_k. Solve the equations for the unique solution.

Step **4**. The corrected trial solution y with evaluated coefficients d_1, d_2, ..., d_k becomes the particular solution y_p.

Symbols

The symbols $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ are reserved for use as arbitrary constants in the general solution $\boldsymbol{y_h}$ of the homogeneous equation.

Symbols $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$, $\ldots$ are reserved for use in the trial solution $\boldsymbol{y}$ of the non-homogeneous equation. Abbreviations: $\boldsymbol{c} =$ constant, $\boldsymbol{d} =$ determined.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Find y_h as a linear combination of atoms computed by applying Euler's theorem to factors of $ar^2 + br + c$.

(b) Apply the **basic trial solution method** to find y_p.

- We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p.
- Calling $d_1, d_2, d_3, \ldots$ *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

Annihilator Polynomial $q(r)$ for $f(x)$

Assume $f(x)$ is a linear combination of atoms. Each atom corresponds to, according to Euler's Theorem, a factor $(r - a - ib)^{k+1}$ of a polynomial. Choose the largest k, over all atoms sharing this factor, and then multiply all maximal factors, taken over all atoms appearing in $f(x)$, to obtain a polynomial $q(r)$. If a complex factor appears in $q(r)$, then multiply $q(r)$ by the conjugate of that factor. Finally, $q(r)$ is real. It is called an *annihilator polynomial* for $f(x)$.

Example: If $f(x) = x + x^2 + \cos x$, then the maximal factors according to Euler's Theorem are r^3 and $(r - i)$. Then $q(r) = r^3(r - i)(r + i)$, because complex factors must appear with their conjugates. Finally, $q(r) = r^3(r^2 + 1)$.

Annihilator Differential Equation $Aw = 0$ for $f(x)$

Assume $f(x)$ is a linear combination of atoms and $q(r)$ is an annihilator polynomial for $f(x)$. Then $q(r)$ is the characteristic polynomial of a higher order linear homogeneous equation $Aw = 0$ and $w = f(x)$ is a particular solution of this differential equation. The equation $Aw = 0$ is called an *annihilator differential equation* for $f(x)$.

Example: If $f(x) = x + x^2 + \cos x$, then $q(r) = r^3(r^2 + 1) = r^5 + r^3$ and $Aw = 0$ is the differential equation $w^{(5)} + w^{(3)} = 0$. We verify that $f(x)$ satisfies

$$\begin{aligned} A(f) &= f^{(5)} + f^{(3)} \\ &= ((x + x^2)^{(5)} + (x + x^2)^{(3)}) + ((\cos x)^{(5)} + (\cos x)^{(3)}) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Correction rule I: The Annihilator Method

The rule computes the corrected trial solution y without having to substitute y into the non-homogeneous differential equation $Ly = f$.

- Let $p(r)$ be the characteristic polynomial for the homogeneous differential equation $Ly = 0$, from which we obtain the homogeneous general solution $y_h(x)$.
- Let $q(r)$ be an annihilator polynomial for $f(x)$ and $Aw = 0$ its annihilator differential equation, so that $A(f) = 0$. *We never need to find* $Aw = 0$ *explicitly*!
- Multiply $p(r)$ and $q(r)$ to obtain $p(r)q(r)$, which is the characteristic equation of $A(Ly) = 0$. Then $y(x)$ is a solution of $A(Ly) = 0$, because $A(Ly) = A(f) = 0$. Expand y as a linear combination of atoms selected from the maximal factors of $p(r)q(r)$.
- The superposition principle $y(x) = y_h(x) + y_p(x)$ implies the corrected trial solution $y(x)$ is obtained by removal of all atoms shared with $y_h(x)$.

Correction rule II

The rule predicts the corrected trial solution y without having to substitute y into the differential equation.

- Write down y_h, the general solution of homogeneous equation $ay'' + by' + cy = 0$, having arbitrary constants c_1, c_2. Create the corrected trial solution y iteratively, as follows.
- Cycle through each term $d_p A$, where A is a atom. If A is also an atom appearing in y_h, then multiply $d_p A$ and each **related atom** term $d_q B$ by x. Other terms appearing in y are unchanged.
- Repeat until each term $d_p A$ has atom A distinct from all atoms appearing in homogeneous solution y_h. The modified expression y is called the **corrected trial solution**.

Correction rule III

The rule predicts the corrected trial solution y without substituting it into the differential equation. This iterative algebraic method uses the atom list of the homogeneous equation to create y.

- Write down the roots of the characteristic equation. Let L denote the list of distinct atoms for these roots.
- Cycle through each term d_pA, where A is a atom. If A appears in list L, then multiply d_pA and each **related atom** term d_qB by x. Other terms appearing in y are unchanged.
- Repeat until the atom A in an arbitrary term d_pA of y does not appear in list L.[a] The modified expression y is called the **corrected trial solution**.

[a]The number s of repeats for initial term d_pA equals the multiplicity of the root r which created atom A in list L.

Definition of function atomRoot

- $\mathbf{atomRoot}(x^j e^{rx}) = r$ for r real.
- $\mathbf{atomRoot}(x^j e^{ax} \cos bx) = \mathbf{atomRoot}(x^j e^{ax} \sin bx) = a + ib$.

Correction rule IV

The rule predicts the corrected trial solution y without substituting it into the differential equation. This algebraic method uses the roots of the characteristic equation to correct y.

- Write down the roots of the characteristic equation as a list R, according to multiplicity.
- Subdivide trial solution y into groups G of related atoms, by collecting terms and inserting parentheses.
- If a group G contains an atom A with $r = \mathbf{atomRoot}(A)$ in list R, then multiply all terms of G by x^s, where s is the multiplicity of root r.
- Repeat the previous step for all groups G in y. The modified expression y is called the **corrected trial solution**.