Chapter 5

Second Order Linear Equations

Studied here are linear differential equations of the second order

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

Important to the theory is continuity of the **coefficients** $a(x)$, $b(x)$, $c(x)$ and the **non-homogeneous term** $f(x)$, also called the **forcing term** or the **input**.

5.1 Linear Constant Equations

Studied is the equation

$$ay'' + by' + cy = 0$$

where $a \neq 0$, b and c are constants. An explicit formula for the general solution is developed. Prerequisites are the quadratic formula, complex numbers, Cramer's rule for 2×2 linear systems and first order linear differential equations.

Theorem 1 (Recipe for Constant Equations)
Let $a \neq 0$, b and c be real constants. Let r_1, r_2 be the two roots of $ar^2 + br + c = 0$, real or complex. If complex, then let $r_1 = \overline{r_2} = \alpha + i\beta$ with $\beta > 0$. Consider the following three cases, organized by the sign of the discriminant $\mathcal{D} = b^2 - 4ac$:

$\mathcal{D} > 0$ (Real distinct roots)	$y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$.
$\mathcal{D} = 0$ (Real equal roots)	$y_1 = e^{r_1 x}$, $y_2 = xe^{r_1 x}$.
$\mathcal{D} < 0$ (Conjugate roots)	$y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$.

Then y_1, y_2 are two solutions of $ay'' + by' + cy = 0$ and all solutions are given by $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitrary constants.

The proof appears on page 193. Examples 1–3, page 191, cover the three cases.

A **general solution** is an expression that represents all solutions of the differential equation. Theorem 1 gives an expression of the form

$$y = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are *arbitrary* constants and y_1, y_2 are special solutions of the differential equation, determined by the roots of the **characteristic equation** $ar^2 + br + c = 0$. The terminology **recipe** means that the general solution can be written out at very high speed with no justification required.

The **initial value problem** for $ay'' + by' + cy = 0$ selects the constants c_1, c_2 in the general solution $y = c_1 y_2 + c_2 y_2$ from **initial conditions** of the form $y(x_0) = d_1$, $y'(x_0) = d_2$. In these conditions, x_0 is a given point in $-\infty < x < \infty$ and d_1, d_2 are two real numbers.

Theorem 2 (Uniqueness)
Let $a \neq 0$, b, c, x_0, d_1 and d_2 be constants. The initial value problem $ay'' + by' + cy = 0$, $y(x_0) = d_1$, $y'(x_0) = d_2$ has one and only one solution, found from the general solution $y = c_1 y_1 + c_2 y_2$ by applying Cramer's rule or the method of elimination.

The proof appears on page 194. For Cramer's rule details, see Example 4, page 192.

The two theorems taken together give a *working rule* for solving a linear constant equation:

> To solve $ay'' + by' + cy = 0$, find the roots of the characteristic equation $ar^2 + br + c = 0$ and then apply the recipe to write down y_1, y_2. The general solution is then $y = c_1 y_1 + c_2 y_2$. If initial conditions are given, then determine c_1, c_2 explicitly, otherwise c_1, c_2 remain arbitrary.

Theorem 3 (Superposition)
Let $a \neq 0$, b and c be constants. Assume y_1, y_2 are solutions of $ay'' + by' + cy = 0$ and c_1, c_2 are constants. Then $y = c_1 y_1 + c_2 y_2$ is a solution of $ay'' + by' + cy = 0$.

A proof appears on page 194. The result is implicitly used in Theorem 1, in order to show that a general solution satisfies the differential equation.

Recipe Speed. The time taken to write out the general solution varies among individuals and according to the algebraic complexity of the characteristic equation. Judge your understanding of the *recipe* by these statistics: most persons can write out the general solution in under 60 seconds. Especially simple equations like $y'' = 0$, $y''+y = 0$, $y''-y = 0$, $y'' + 2y' + y = 0$, $y'' + 3y' + 2y = 0$ are finished in less than 30 seconds.

Graphics. Computer programs can produce plots for initial value problems. They cannot plot **symbolic solutions** containing the arbitrary variables c_1, c_2 that appear in the general solution.

Recipe Errors. Below in Table 1 are recorded some common but fatal errors made in writing out the general solution.

Table 1. Errors in Applying the Constant Equation Recipe.

Bad equation	For $y'' - y = 0$, the correct characteristic equation is $r^2 - 1 = 0$. Commonly, $r^2 - r = 0$ is written, an error.
Sign reversal	For factored equation $(r + 1)(r + 2) = 0$, the roots are $r = -1$, $r = -2$. A common error is to claim $r = 1$ is a root.
Miscopy signs	The equation $r^2+2r+2 = 0$ has complex conjugate roots $\alpha \pm \beta i$, where $\alpha = -1$ and $\beta = 1$ ($\beta > 0$ is required). A common error is to miscopy signs on α and/or β.
Copying $\pm i$	The equation $r^2 + 4 = 0$ has roots $\alpha \pm \beta i$ where $\alpha = 0$ and $\beta = 2$. A common mistake is to report "solutions" $\cos(\pm 2ix)$ and $\sin(\pm 2ix)$ – neither $\pm$ nor the complex unit i should be copied.

1 Example (Case 1) Solve $y'' + y' - 2y = 0$.

Solution: The general solution is $y = c_1e^x +c_2e^{-2x}$. Ordering is not important; an equivalent answer is $y = c_1e^{-2x} + c_2e^x$. The answer will be justified below, by finding y_1, y_2 in the *recipe*.

The characteristic equation $r^2 + r - 2 = 0$ is found formally by replacements $y'' \to r^2$, $y' \to r$ and $y \to 1$ in the differential equation. Formal replacement reduces errors.

A college algebra method called *inverse-FOIL* applies to factor $r^2 + r - 2 = 0$ into $(r - 1)(r + 2) = 0$. The roots are $r = 1$, $r = -2$.

Applying case $\mathcal{D} > 0$ of the *recipe* gives solutions $y_1 = e^x$ and $y_2 = e^{-2x}$. If the roots are listed in reverse order, then the form of the answer will change to the equivalent one reported above.

2 Example (Case 2) Solve $4y'' + 4y' + y = 0$.

Solution: The general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$. To justify this formula, find the characteristic equation $4r^2 + 4r + 1 = 0$ and factor it by the *inverse-FOIL method* or *square completion* to obtain $(2r+1)^2 = 0$. The roots are both $-1/2$.

Case $\mathcal{D} = 0$ of the *recipe* gives $y_1 = e^{-x/2}$, $y_2 = xe^{-x/2}$. Then the general solution is $y = c_1 y_1 + c_2 y_2$, which completes the verification.

3 Example (Case 3) Solve $4y'' + 2y' + y = 0$.

Solution: The solution is $y = c_1 e^{-x/4}\cos(\sqrt{3}x/4) + c_2 e^{-x/4}\sin(\sqrt{3}x/4)$. This formula is justified below, by showing that the solutions y_1, y_2 of the *recipe* are given by $y_1 = e^{-x/4}\cos(\sqrt{3}x/4)$ and $y_2 = e^{-x/4}\sin(\sqrt{3}x/4)$.

The characteristic equation is $4r^2 + 2r + 1 = 0$. The roots by the *quadratic formula* are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ College algebra formula for the roots of the quadratic $ar^2 + br + c = 0$.

$$= \frac{-2 \pm \sqrt{2^2 - (4)(4)(1)}}{(2)(4)}$$ Substitute $a = 4$, $b = 2$, $c = 1$.

$$= -\frac{1}{4} \pm \frac{\sqrt{-1}\sqrt{12}}{8}$$ Simplify. Used $\sqrt{(-1)(12)} = \sqrt{-1}\sqrt{12}$.

$$= -\frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$ Convert to complex form, $i = \sqrt{-1}$.

The real part of the root is labeled $\alpha = -1/4$. The two imaginary parts are $\sqrt{3}/4$ and $-\sqrt{3}/4$. Only the positive one is labeled, the other being discarded: $\beta = \sqrt{3}/4$.

The *recipe* case $\mathcal{D} < 0$ applies to give solutions $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. Substitution of $\alpha = -1/4$ and $\beta = \sqrt{3}/4$ results in the formulas $y_1 = e^{-x/4}\cos(\sqrt{3}x/4)$, $y_2 = e^{-x/4}\sin(\sqrt{3}x/4)$. The verification is complete.

4 Example (Initial Value Problem) Solve $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = -2$ and graph the solution on $0 \le x \le 2$.

Solution: The solution to the initial value problem is $y = e^{-2x}$. The graph appears in Figure 1. Justification and graph construction appear below.

The general solution is $y = c_1 e^x + c_2 e^{-2x}$, from Example 1. The problem of finding c_1, c_2 uses the two equations $y(0) = 1$, $y'(0) = -2$ and the general solution to obtain expanded equations for c_1, c_2:

$$\begin{array}{rcrcr} e^0 c_1 & + & e^0 c_2 & = & 1, \\ e^0 c_1 & - & 2e^0 c_2 & = & -2. \end{array}$$

The equations will be solved by the method of elimination. Since $e^0 = 1$, the equations are subject to simplification. Subtracting them eliminates the variable c_1 to give $3c_2 = 3$. Therefore, $c_2 = 1$ and back-substitution finds $c_1 = 0$. Then $y = c_1 e^x + c_2 e^{-2x}$ reduces to $y = e^{-2x}$.

To graph the solution is a routine application of curve library methods, so no hand-graphing methods will be discussed. To produce a computer graphic of the solution, the following code is offered.

```
plot(exp(-2*x),x=0..2);          Maple V
Plot[{exp(-2 x)},{x,0,2}];       Mathematica
plot [0:2] exp(-2*x)             Gnuplot
x=0:0.05:2; plot(x,exp(-2*x))    Matlab and Scilab
```

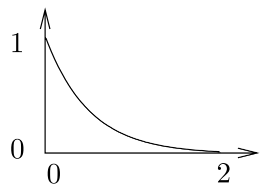

Figure 1. Exponential solution $y = e^{-2x}$.

Proof of Theorem 1: To show that y_1 and y_2 are solutions is left to the exercises. For the remainder of the proof, assume y is a solution of $ay'' + by' + cy = 0$. It has to be shown that $y = c_1 y_1 + c_2 y_2$ for some real constants c_1, c_2.

Algebra background. In college algebra it is shown that the polynomial $ar^2 + br + c$ can be written in terms of its roots r_1, r_2 as $a(r - r_1)(r - r_2)$. In particular, the sum and product of the roots satisfy the relations $b/a = -r_1 - r_2$ and $c/a = r_1 r_2$.

Case $\mathcal{D} > 0$. The equation $ay'' + by' + cy = 0$ can be re-written in the form $y'' - (r_1 + r_2)y' + r_1 r_2 y = 0$ due to the college algebra relations for the sum and product of the roots of a quadratic equation. The equation "factors" into $(y' - r_2 y)' - r_1(y' - r_2 y) = 0$ which suggests the substitution $u = y' - r_2 y$. Then $ay'' + by' + cy = 0$ is equivalent to the first order system

$$\begin{aligned} u' - r_1 u &= 0, \\ y' - r_2 y &= u. \end{aligned}$$

Growth-decay theory, page 3, applied to the first equation gives $u = u_0 e^{r_1 x}$. The second equation $y' - r_2 y = u$ is then solved by the integrating factor method, as in Example 11, page 75. This gives $y = y_0 e^{r_2 x} + u_0 e^{r_1 x}/(r_1 - r_2)$. Therefore, any possible solution y has the form $c_1 e^{r_1 x} + c_2 e^{r_2 x}$ for some c_1, c_2. This completes the proof of the case $\mathcal{D} > 0$.

Case $\mathcal{D} = 0$. The details follow the case $\mathcal{D} > 0$, except that $y' - r_2 y = u$ has a different solution, $y = y_0 e^{r_1 x} + u_0 x e^{r_1 x}$ (exponential factors $e^{r_1 x}$ and $e^{r_2 x}$ cancel because $r_1 = r_2$). Therefore, any possible solution y has the form $c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ for some c_1, c_2. This completes the proof of the case $\mathcal{D} = 0$.

Case $\mathcal{D} < 0$. The equation $ay'' + by' + cy = 0$ can be re-written in the form $y'' - (r_1 + r_2)y' + r_1 r_2 y = 0$ as in the case $\mathcal{D} > 0$, even though y is real and the roots are complex. The substitution $u = y' - r_2 y$ gives the same equivalent system as in the case $\mathcal{D} > 0$. The solutions are symbolically the same, $u = u_0 e^{r_1 x}$ and $y = y_0 e^{r_1 x} + u_0 e^{r_1 x}/(r_1 - r_2)$. Therefore, any possible real solution y has the form $C_1 e^{r_1 x} + C_2 e^{r_2 x}$ for some possibly complex C_1, C_2. Taking the real part of both sides of this equation gives $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$ for some real constants c_1, c_2, as follows:

$y = \mathcal{R}\mathrm{e}(y)$	Because y is real.
$= \mathcal{R}\mathrm{e}(C_1 e^{r_1 x} + C_2 e^{r_2 x})$	Substitute.
$= e^{\alpha x}\,\mathcal{R}\mathrm{e}(C_1 e^{i\beta x} + C_2 e^{-i\beta x})$	Use $e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x}$.
$= e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$	Where $c_1 = \mathcal{R}\mathrm{e}(C_1 + C_2)$ and $c_2 = \mathcal{I}\mathrm{m}(C_2 - C_1)$ are real. Applied $e^{i\beta} = \cos\beta + i\sin\beta$.

This completes the proof of the case $\mathcal{D} < 0$.

Proof of Theorem 2: The left sides of the two requirements $y(x_0) = d_1$, $y'(x_0) = d_2$ are expanded using the relation $y = c_1y_1 + c_2y_2$ to obtain the following system of equations for the unknowns c_1, c_2:

$$\begin{array}{rcrcl} y_1(x_0)c_1 & + & y_2(x_0)c_2 & = & d_1, \\ y_1'(x_0)c_1 & + & y_2'(x_0)c_2 & = & d_2. \end{array}$$

If the determinant of coefficients

$$\Delta = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$

is nonzero, then Cramer's rule says that the solutions c_1, c_2 are given as quotients

$$c_1 = \frac{d_1y_2'(x_0) - d_2y_2(x_0)}{\Delta}, \quad c_2 = \frac{y_1(x_0)d_2 - y_1'(x_0)d_1}{\Delta}.$$

The organization of the proof is made from the three cases of the *recipe*, using x instead of x_0, to simplify notation. The issue of a unique solution has now reduced to verification of $\Delta \neq 0$, in the three cases.

Case $\mathcal{D} > 0$. Then

$$\begin{aligned} \Delta &= e^{r_1x}r_2e^{r_2x} - r_1e^{r_1x}e^{r_2x} && \text{Substitute for } y_1,\, y_2. \\ &= (r_2 - r_1)e^{r_1x+r_2x} && \text{Simplify.} \\ &\neq 0 && \text{Because } r_1 \neq r_2. \end{aligned}$$

Case $\mathcal{D} = 0$. Then

$$\begin{aligned} \Delta &= e^{r_1x}(e^{r_1x} + r_1xe^{r_1x}) - r_1e^{r_1x}xe^{r_1x} && \text{Substitute for } y_1,\, y_2. \\ &= e^{2r_1x} && \text{Simplify.} \\ &\neq 0 \end{aligned}$$

Case $\mathcal{D} < 0$. Then $r_1 = \overline{r_2} = \alpha + i\beta$ and

$$\begin{aligned} \Delta &= \beta e^{2\alpha x}(\cos^2\beta x + \sin^2\beta x) && \text{Cancel } \alpha e^{2\alpha x}\sin(\beta x)\cos(\beta x). \\ &= \beta e^{2\alpha x} && \text{Trigonometric identity.} \\ &\neq 0 && \text{Because } \beta > 0. \end{aligned}$$

In applications, the more efficient method of elimination is used to find c_1, c_2. In some references, it is called *Gaussian elimination.*

Proof of Theorem 3: The three terms of the differential equation are computed using the expression $y = c_1y_1 + c_2y_2$:

$$\begin{array}{ll} \text{Term 1:} & cy = cc_1y_1 + cc_2y_2 \\ \text{Term 2:} & by' = b(c_1y_1 + c_2y_2)' \\ & = bc_1y_1' + bc_2y_2' \\ \text{Term 3:} & ay'' = a(c_1y_1 + c_2y_2)'' \\ & = ac_1y_1'' + ac_2y_2'' \end{array}$$

The left side LHS of the differential equation is the sum of the three terms. It is simplified as follows:

LHS $= c_1[ay_1'' + by_1' + cy_1]$	Add terms 1,2 and 3,
$\quad + c_2[ay_2'' + by_2' + cy_2]$	then collect on c_1, c_2.
$= c_1[0] + c_2[0]$	Both y_1, y_2 satisfy $ay'' + by' + cy = 0$.
= RHS	The left and right sides match.

Exercises 5.1

Recipe General Solution. Apply the recipe for constant equations, Theorem 1, page 189, to write out the general solution. Model the solution after Examples 1–3, page 191.

1. $y'' = 0$

2. $3y'' = 0$

3. $y'' + y' = 0$

4. $3y'' + y' = 0$

5. $y'' + 3y' + 2y = 0$

6. $y'' - 3y' + 2y = 0$

7. $y'' - y' - 2y = 0$

8. $y'' - 2y' - 3y = 0$

9. $y'' + y = 0$

10. $y'' + 4y = 0$

11. $y'' + 16y = 0$

12. $y'' + 8y = 0$

13. $y'' + y' + y = 0$

14. $y'' + y' + 2y = 0$

15. $y'' + 2y' + y = 0$

16. $y'' + 4y' + 4y = 0$

17. $3y'' + y' + y = 0$

18. $9y'' + y' + y = 0$

19. $5y'' + 25y' = 0$

20. $25y'' + y' = 0$

21. **(Recipe case 1)** Let $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$. Assume factorization $ar^2 + br + c = a(r - r_1)(r - r_2)$. Show that y_1, y_2 are solutions of $ay'' + by' + cy = 0$.

22. **(Recipe case 2)** Let $y_1 = e^{r_1 x}$, $y_2 = x\,e^{r_1 x}$. Assume factorization $ar^2 + br + c = a(r - r_1)(r - r_1)$. Show that y_1, y_2 are solutions of $ay'' + by' + cy = 0$.

23. **(Recipe case 3)** Let $y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$, with $\beta > 0$. Assume factorization $ar^2 + br + c = a(r - \alpha - i\beta)(r - \alpha + i\beta)$. Show that y_1, y_2 are solutions of $ay'' + by' + cy = 0$.