2.3 Linear Equations

Definition 3 (Linear Differential Equation)
An equation $y' = f(x, y)$ is called **first-order linear** or a **linear equation** provided functions $p(x)$ and $r(x)$ can be defined to re-write the equation in the **standard form**

$$y' + p(x)y = r(x). \tag{1}$$

In most applications, p and r are assumed to be continuous. Function $p(x)$ is called the **coefficient of** y. Function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

In examples, a linear equation is identified by matching:

$$\frac{dy}{dx} + \begin{pmatrix} p(x)\text{, an expression} \\ \text{independent of } y \end{pmatrix} y = \begin{pmatrix} r(x)\text{, another expression} \\ \text{independent of } y \end{pmatrix}$$

.

Calculus Test:

> An equation $y' = f(x, y)$ with f continuously differentiable is **linear** provided $\dfrac{\partial f(x,y)}{\partial y}$ is independent of y.

If the test is passed, then standard linear form (1) is obtained by defining $r(x) = f(x, 0)$ and $p(x) = -\partial f/\partial y(x, y)$. Readers should pause and verify this statement.

Key Examples

$L\dfrac{dI}{dt} + RI = E$	The LR-circuit equation. Symbols L, R and E are respectively inductance, resistance and electromotive force, while $I(t)$ = current in amperes and t = time. $\boxed{1}$
$\dfrac{du}{dt} = -h(u - u_1)$	Newton's cooling equation. In the roast model, the oven temperature is u_1 and the meat thermometer reading is $u(t)$, with t = time. $\boxed{2}$

Notes.
$\boxed{1}$ Linear equation $y' + p(x)y = r(x)$ is realized with symbols y, x, p, r undergoing name changes. Define $x = t$, $y = I$, $p(x) = R/L$, $r(x) = E/L$.
$\boxed{2}$ Linear equation $y' + p(x)y = r(x)$ is realized by re-defining symbols y, x, p, r. Start with the equation re-arranged algebraically to $\dfrac{du}{dt} + hu = hu_1$. Define $x = t$, $y = u$, $p(x) = h$, $r(x) = hu_1$.

Homogeneous Equation $y' + p(x)y = 0$

Homogeneous equations $y' + p(x)y = 0$ occur in applications devoid of external forces, like an LR-circuit with no battery in the circuit. Justified later is the fundamental result for such systems:

The general solution of $\dfrac{dy}{dx} + p(x)y = 0$ is the fraction

$$y(x) = \frac{\text{constant}}{\text{integrating factor}} = \frac{c}{W(x)}$$

where **integrating factor** $W(x)$ is defined by the equation

$$W(x) = e^{\int p(x)dx}.$$

An Illustration. The LR-circuit equation $\dfrac{dI}{dt} + 2I = 0$ is the model equation $y' + p(x)y = 0$ with $p(x) = 2$. Then $W(x) = e^{\int 2dx} = e^{2x}$, with integration constant set to zero. The general solution of $y' + 2y = 0$ is given by

$$y = \frac{c}{W(x)} = \frac{c}{e^{2x}} = ce^{-2x}.$$

The current is $I(t) = c\,e^{-2t}$, by the variable swap $x \to t$, $y \to I$.

Definition 4 (Integrating Factor)
An **integrating factor** $W(x)$ for equation $y' + p(x)y = r(x)$ is

$$W(x) = e^{\int p(x)dx}.$$

Lemma 1 (Integrating Factor Identity) The integrating factor $W(x)$ satisfies the differential equation

$$W'(x) = p(x)W(x).$$

Details. Write $W = e^u$ where $u = \int p(x)dx$. By the fundamental theorem of calculus, $u' = p(x) =$ the integrand. Then the chain rule implies $w' = u'e^u = u'W = pW$.

A Shortcut. Factor $W(x)$ is generally expressed as a simplified expression, with integration constant set to zero and absolute value symbols removed. See the exercises for details about this simplification. For instance, integration in the special case $p(x) = 2$ formally gives $\int p(x)dx = \int 2dx = 2x + c_1$. Then the integrating factor becomes $W(x) = e^{\int 2dx} = e^{2x+c_1} = e^{2x}e^{c_1}$. Fraction $c/W(x)$ equals c_2/e^{2x}, where $c_2 = c/e^{c_1}$. The lesson is that we could have chosen $c_1 = 0$ to produce the same fraction. This is a shortcut, recognized as such, but we use it in examples to save effort.

Non-Homogeneous Equation $y' + p(x)y = r(x)$

Definition 5 (Homogeneous and Particular Solution)
Let $W(x)$ be an integrating factor constructed for $y'+p(x)y = r(x)$, that is, $W(x) = e^u$, where $u = \int p(x)dx$ is an antiderivative of $p(x)$.

Symbol y_h, called the **homogeneous solution**, is defined by the expression

$$y_h(x) = \frac{c}{W(x)}.$$

Symbol y_p, called a **particular solution**, is defined by the expression

$$y_p(x) = \frac{\int r(x)W(x)dx}{W(x)}$$

Theorem 5 (Homogeneous and Particular Solutions)
(a) Expression $y_h(x)$ is a solution of the homogeneous differential equation $y' + p(x)y = 0$.

(b) Expression $y_p(x)$ is a solution of the non-homogeneous differential equation $y' + p(x)y = r(x)$.

Proof:
(a) Define $y = c/W$. We prove $y' + py = 0$. Formula $y = c/W$ implies $(yW)' = (c)' = 0$. The product rule and the Lemma imply $(yW)' = y'W + yW' = y'W + y(pW) = (y' + py)W$. Then $(yW)' = 0$ implies $y' + py = 0$. The proof is complete.

(b) We prove $y' + py = r$ when y is replaced by the fraction y_p. Define $C(x) = \int r(x)W(x)dx$, so that $y = C(x)/W(x)$. The fundamental theorem of calculus implies $C'(x) = r(x)W(x)$. The product rule and the Lemma imply $C' = (yW)' = y'W + yW' = y'W + ypW = (y' + py)W$. Competition between the two equations for C' gives $rW = (y' + py)W)$. Cancel W to obtain $r = y' + py$. The proof is complete.

Historical Note. The formula for $y_p(x)$ has the historical name **variation of constants** or **variation of parameters**. Both y_h and y_p have the same form C/W, with $C(x)$ constant for y_h and $C(x)$ equal to a function of x for y_p: **variation of constant** c in y_h produces the expression for y_p.

Experimental Viewpoint. The particular solution y_p depends on the forcing term $r(x)$, but the homogeneous solution y_h does not. Experimentalists view the computation of y_p as a *single experiment* in which the state y_p is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$. This particular experimental solution y_p^* is given by the definite integral formula

$$y_p^*(x) = \frac{1}{W(x)} \int_{x_0}^{x} r(x)W(x)dx. \tag{2}$$

Superposition. The sum of constant multiples of solutions to $y' + p(x)y = 0$ is again a solution. The next two theorems are **superposition** for $y' + p(x)y = r(x)$.

Theorem 6 (General Solution = Homogeneous + Particular)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p$.

In short, a linear equation has the solution structure *homogeneous plus particular*.

The constant c in formula y_h and the integration constant in $\int W(x)rx)dx$ can always be selected to satisfy initial condition $y(x_0) = y_0$.

Theorem 7 (Difference of Solutions = Homogeneous Solution)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_1 and y_2 be two solutions of $y' + p(x)y = r(x)$ on $a < x < b$. Then $y = y_1 - y_2$ is a solution of the homogeneous differential equation

$$y' + p(x)y = 0.$$

In short, any two solutions of the non-homogeneous equation differ by some solution y_h of the homogeneous equation.

Integrating Factor Method

The technique called the **method of integrating factors** uses the replacement rule (justified on page 101)

$$\text{(3)} \qquad \text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}.$$

The fraction $(YW)'/W$ is called the **integrating factor fraction**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find W	Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\dfrac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (3).
Method of Quadrature	Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

In identity (3), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. Equation (3) is central to the method, because it collapses the two terms $y'+py$ into a single term $(Wy)'/W$; the method of quadrature applies to $(Wy)' = rW$. The literature calls the exponential factor W an **integrating factor** and equivalence (3) a **factorization** of $Y' + p(x)Y$.

Simplifying an Integrating Factor

Factor W is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $W = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$, because $|x| = (\pm)x$. Let $c_1 = \pm e^C$. Then $W = c_1 W_1$ where $W_1 = x$. The fraction $(Wy)'/W$ reduces to $(W_1 y)'/W_1$, because c_1 cancels. In an application, we choose the simpler expression W_1. The illustration shows that exponentials in W can sometimes be eliminated.

Variation of Constants and $y' + p(x)y = r(x)$

Every solution of $y' + p(x)y = r(x)$ can be expressed as $y = y_h + y_p$, by choosing constants appropriately. The classical **variation of constants formula** puts initial condition zero on y_p and compresses all initial data into the constant c appearing in y_h. The general solution is given by

$$y(x) = \frac{y(x_0)}{W(x)} + \frac{\int_{x_0}^x r(x)W(x)dx}{W(x)}, \quad W(x) = e^{\int_{x_0}^x p(s)ds} \tag{4}$$

Classifying Linear and Non-Linear Equations

Definition 6 (Non-linear Differential Equation)
An equation $y' = f(x, y)$ that fails to be linear is called **non-linear**.

Algebraic Complexity. A linear equation $y' = f(x, y)$ may appear to be non-linear, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer Algebra System. These systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$.

Hand verification can use the same method. To illustrate, consider $y' = f(x, y)$ with $f(x, y) = (x - y)(x + y) + y(y - 2x)$. Compute $f(x, 0) = x^2$, $f_y(x, 0) = -2x$. Because f_y is independent of y, then $y' = f(x, y)$ is the linear equation $y' + p(x)y = r(x)$ with $p(x) = 2x$, $r(x) = x^2$.

Non-Linear Equation Tests. Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x,y)$ depends on y, then $y' = f(x,y)$ is not linear.
2. If $f_{yy}(x,y) \neq 0$, then $y' = f(x,y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

Special Linear Equations

There are fast ways to solve certain linear differential equations that do not employ the linear integrating factor method.

Theorem 8 (Solving a Homogeneous Equation)
Assume $p(x)$ is continuous on $a < x < b$. Then the solution of the homogeneous differential equation $y' + p(x)y = 0$ is given by the formula

$$y(x) = \frac{\text{constant}}{\text{integrating factor}}. \tag{5}$$

Theorem 9 (Solving a Constant-Coefficient Equation)
Assume $p(x)$ and $r(x)$ are constants p, r with $p \neq 0$. Then the solution of the constant-coefficient differential equation $y' + py = r$ is given by the formula

$$\begin{aligned} y(x) &= \frac{\text{constant}}{\text{integrating factor}} + \text{equilibrium solution} \\ &= ce^{-px} + \frac{r}{p}. \end{aligned} \tag{6}$$

Proof: The homogeneous solution is a constant divided by the integrating factor, by Theorem 8. An equilibrium solution can be found by formally setting $y' = 0$, then solving for $y = r/p$. By superposition Theorem 6, the solution y must be the sum of these two solutions. We remark that the case $p = 0$ results in a quadrature equation $y' = r$ which is routinely solved by the method of quadrature.

Examples

12 Example (Shortcut: Homogeneous Equation)
Solve the homogeneous equation $2y' + x^2 y = 0$.

Solution: By Theorem (8), the solution is a constant divided by the integrating factor. First, divide by 2 to get $y' + p(x)y = 0$ with $p(x) = \frac{1}{2}x^2$. Then $\int p(x)dx = x^3/6 + c$ implies $W = e^{x^3/6}$ is an integrating factor. The solution is $y = \dfrac{c}{e^{x^3/6}}$.

13 Example (Shortcut: Constant-Coefficient Equation)
Solve the non-homogeneous constant-coefficient equation $2y' - 5y = -1$.

Solution: The method described here only works for first order constant-coefficient differential equations. If $y' = f(x,y)$ is not linear or it fails to have constant coefficients, then the method fails.

The solution has two steps:

> (1) Find the solution y_h of the homogeneous equation $2y' - 5y = 0$.
> The answer is a constant divided by the integrating factor, which is $y = \dfrac{c}{e^{-5x/2}}$. First divide the equation by 2 to obtain the standard form $y' + (-5/2)y = 0$. Identify $p(x) = -5/2$, then $\int p(x)dx = -5x/2 + c$ and finally $W = e^{-5x/2}$ is the integrating factor. The answer is $y_h = c/W = ce^{5x/2}$.
>
> (2) Find an equilibrium solution y_p for $2y' - 5y = -1$.
> This answer is found by formally replacing y' by zero. Then $y_p = \frac{1}{5}$.

The answer is the sum of the answers from (1) and (2), by superpositon, giving

$$y = y_h + y_p = ce^{5x/2} + \frac{1}{5}.$$

The method of this example is called the **superposition method shortcut**.

14 Example (Integrating Factor Method)

Solve $2y' + 6y = e^{-x}$.

Solution: The solution is $y = \frac{1}{4}e^{-x} + ce^{-3x}$. An answer check appears in Example 16. The details:

$y' + 3y = 0.5e^{-x}$	Divide by 2 to get the standard form.
$W = e^{3x}$	Find the integrating factor $W = e^{\int 3dx}$.
$\dfrac{\left(e^{3x}y\right)'}{e^{3x}} = 0.5e^{-x}$	Replace the LHS of $y' + 3y = 0.5e^{-x}$ by the integrating factor quotient; see page 96.
$\left(e^{3x}y\right)' = 0.5e^{2x}$	Clear fractions. Prepared for quadrature
$e^{3x}y = 0.5\int e^{2x}dx$	Method of quadrature applied.
$y = 0.5\left(e^{2x}/2 + c_1\right)e^{-3x}$	Evaluate the integral. Divide by $W = e^{3x}$.
$= \frac{1}{4}e^{-x} + ce^{-3x}$	Final answer, $c = 0.5c_1$.

15 Example (Superposition)

Find a particular solution of $y' + 2y = 3e^{x}$ with fewest terms.

Solution: The answer is $y = e^{x}$. The first step solves the equation using the integrating factor method, giving $y = e^{x} + ce^{-2x}$; details below. A particular solution with fewest terms, $y = e^{x}$, is found by setting $c = 0$.

Integrating factor method details:

$y' + 2y = 3e^{x}$	The standard form.
$W = e^{2x}$	Find the integrating factor $W = e^{\int 2dx}$.

$$\frac{\left(e^{2x}y\right)'}{e^{3x}} = 3e^x$$ Integrating factor identity applied to $y' + 2y = 3e^x$.

$e^{2x}y = 3\int e^{3x}dx$ Clear fractions and apply quadrature.

$y = \left(e^{3x} + c\right)e^{-2x}$ Evaluate the integral. Isolate y.

$= e^x + ce^{-2x}$ Solution found.

Remarks on Integral Formula (2). Computer algebra systems will compute the solution $y_p^* = e^x - e^{3x_0}e^{-2x}$ of equation (2). It has an extra term because of the condition $y = 0$ at $x = x_0$. The shortest particular solution e^x and the integral formula solution y_p^* differ by a homogeneous solution c_1e^{-2x}, where $c_1 = e^{3x_0}$. To shorten y_p^* to $y_p = e^x$ requires knowing the homogeneous solution, then apply superposition $y = y_p + y_h$ to extract a particular solution.

16 Example (Answer Check) Show the answer check details for $2y' + 6y = e^{-x}$ and candidate solution $y = \frac{1}{4}e^{-x} + ce^{-3x}$.

Solution: **Details**:

LHS $= 2y' + 6y$	Left side of the equation $2y' + 6y = e^{-x}$.
$= 2(-\frac{1}{4}e^{-x} - 3ce^{-3x}) + 6(\frac{1}{4}e^{-x} + ce^{-3x})$	Substitute for y.
$= e^{-x} + 0$	Simplify terms.
$=$ RHS	DE verified.

17 Example (Finding y_h and y_p)
Find the homogeneous solution y_h and a particular solution y_p for the equation $2xy' + y = 4x^2$ on $x > 0$.

Solution: The solution by the integrating factor method is $y = 0.8x^2 + cx^{-1/2}$; details below. Then $y_h = cx^{-1/2}$ and $y_p = 0.8x^2$ give $y = y_h + y_p$.

The symbol y_p stands for *any* particular solution. It should be free of any arbitrary constants c.

Integral formula (2) gives a *different* particular solution $y_p^* = 0.8x^2 - 0.8x_0^{5/2}x^{-1/2}$. It differs from the shortest particular solution $0.8x^2$ by a homogeneous solution $Kx^{-1/2}$.

Integrating factor method details:

$y' + 0.5y/x = 2x$	Standard form. Divided by $2x$.
$p(x) = 0.5/x$	Identify coefficient of y. Then $\int p(x)dx = 0.5\ln\lvert x\rvert + c$.
$W = e^{0.5\ln\lvert x\rvert + c}$	The integrating factor is $W = e^{\int p}$.
$W = e^{0.5\ln\lvert x\rvert}$	Choose integration constant zero.
$= \lvert x\rvert^{1/2}$	Used $\ln u^n = n\ln u$. Simplified W found.
$\frac{\left(x^{1/2}y\right)'}{x^{1/2}} = 2x$	Integrating factor identity applied on the left. Assumed $x > 0$.

$x^{1/2}y = 2\int x^{3/2}dx$	Clear fractions. Apply quadrature.
$y = \left(4x^{5/2}/5 + c\right)x^{-1/2}$	Evaluate the integral. Divide to isolate y.
$\quad = \frac{4}{5}x^2 + cx^{-1/2}$	Solution found.

18 Example (Classification) Classify the equation $y' = x + \ln(xe^y)$ as linear or non-linear.

Solution: It's linear, with standard linear form $y' + (-1)y = x + \ln x$. To explain why, the term $\ln(xe^y)$ on the right expands into $\ln x + \ln e^y$, which in turn is $\ln x + y$, using logarithm rules. Because $e^y > 0$, then $\ln(xe^y)$ makes sense for only $x > 0$. Henceforth, assume $x > 0$.

Computer algebra test $f(x,y) = f(x,0) + f_y(x,0)y$. Expected is LHS − RHS = 0 after simplification. This example produced $\ln e^y - y$ instead of 0, evidence that limitations may exist.

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assume(x>0):
f:=(x,y)->x+ln(x*exp(y)):
LHS:=f(x,y):
RHS:=f(x,0)+subs(y=0,diff(f(x,y),y))*y:
simplify(LHS-RHS);
```

If the test *passes*, then $y' = f(x,y)$ becomes $y' = f(x,0) + f_y(x,0)y$. This example gives $y' = x + \ln x + y$, which converts to the standard linear form $y' + (-1)y = x + \ln x$.

Details and Proofs

Justification of Factorization (3): It is assumed that $Y(x)$ is a given but otherwise arbitrary differentiable function. Equation (3) will be justified in its fraction-free form

$$\text{(7)} \qquad \left(Ye^{\mathbf{P}}\right)' = \left(Y' + pY\right)e^{\mathbf{P}}, \quad \mathbf{P}(x) = \int p(x)dx.$$

LHS $= \left(Ye^{\mathbf{P}}\right)'$	The left side of equation (7).
$\quad = Y'e^{\mathbf{P}} + \left(e^{\mathbf{P}}\right)'Y$	Apply the product rule $(uv)' = u'v + uv'$.
$\quad = Y'e^{\mathbf{P}} + pe^{\mathbf{P}}Y$	Use the chain rule $(e^u)' = u'e^u$ and $\mathbf{P}' = p$.
$\quad = (Y' + pY)e^{\mathbf{P}}$	The common factor is $e^{\mathbf{P}}$.
$\quad =$ RHS	The right hand side of equation (7).

Justification of Formula (4):

Existence. Because the formula is $y = y_h + y_p$ for particular values of c and the constant of integration, then y is a solution by superposition Theorem (6) and existence Theorem (5).

Uniqueness. It remains to show that the solution given by (4) is the only solution. Start by assuming Y is another, subtract them to obtain $u = y - Y$. Then $u' + pu = 0$, $u(x_0) = 0$. To show $y \equiv Y$, it suffices to show $u \equiv 0$.

According to the integrating factor method, the equation $u' + pu = 0$ is equivalent to $(uW)' = 0$. Integrate $(uW)' = 0$ from x_0 to x, giving $u(x)W(x) = u(x_0)W(x_0)$. Since $u(x_0) = 0$ and $W(x) \neq 0$, it follows that $u(x) = 0$ for all x. This completes the proof.

Remarks on Picard's Theorem. The Picard-Lindelöf theorem, page 67, implies existence-uniqueness, but only on a smaller interval, and furthermore it supplies no practical formula for the solution. Formula (4) is therefore an improvement over the results obtainable from the general theory.

Exercises 2.3

Integrating Factor Method. Apply the integrating factor method, page 96, to solve the given linear equation. See the examples starting on page 99 for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

20. $2y' + y = 2e^{-x}$

Superposition. Find a particular solution with fewest terms. See Example 15, page 99.

21. $3y' = x$

22. $3y' = 2x$

23. $y' + y = 1$

24. $y' + 2y = 2$

25. $2y' + y = 1$

26. $3y' + 2y = 1$

27. $y' - y = e^{x}$

28. $y' - y = xe^{x}$

29. $xy' + y = \sin x$ $(x > 0)$

30. $xy' + y = \cos x$ $(x > 0)$

31. $y' + y = x - x^2$

32. $y' + y = x + x^2$

General Solution. Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 17, page 100.

33. $y' + y = 1$

34. $xy' + y = 2$

35. $y' + y = x$

36. $xy' + y = 2x$

37. $y' - y = x + 1$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2$ $(x > 0)$

40. $xy' + y = 2x^2$ $(x > 0)$

Classification. Classify as linear or non-linear. Use the test $f(x, y) = f(x, 0) + f_y(x, 0)y$ and a computer algebra system, when available, to check the answer. See Example 18, page 101.

41. $y' = 1 + 2y^2$

42. $y' = 1 + 2y^3$

43. $yy' = (1 + x)\ln e^y$

44. $yy' = (1 + x)\left(\ln e^y\right)^2$

45. $y' \sec^2 y = 1 + \tan^2 y$

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1 + y) = xy$

48. $y' = y(1 + y)$

49. $xy' = (x + 1)y - xe^{\ln y}$

50. $2xy' = (2x + 1)y - xye^{-\ln y}$

Shortcuts. Apply theorems for the homogeneous equation $y' + p(x)y = 0$ or for constant coefficient equations $y' + py = r$. Solutions should be done without paper or pencil, then write the answer and check it.

51. $y' - 5y = -1$

52. $3y' - 5y = -1$

53. $2y' + xy = 0$

54. $3y' - x^2y = 0$

55. $y' = 3x^4y$

56. $y' = (1 + x^2)y$

57. $\pi y' - \pi^2 y = -e^2$

58. $e^2y' + e^3y = \pi^2$

59. $xy' = (1 + x^2)y$

60. $e^xy' = (1 + e^{2x})y$

Proofs and Details.

61. Prove directly without appeal to Theorem 6 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

62. Prove that y_p^* given by equation (2) and $y_p = W^{-1}\int r(x)W(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

63. The equation $y' = r$ with r constant can be solved by quadrature, without pencil and paper. Find y.

64. The equation $y' = r(x)$ with $r(x)$ continuous can be solved by quadrature. Find a formula for y.