Name ___________________________ Class Time _____

Math 2250 Extra Credit Problems
Chapter 8
S2015

Submitted work. Please submit one stapled package with this sheet on top. Kindly check-mark the problems submitted and label the paper **Extra Credit**. Label each solved problem with its corresponding problem number, e.g., Xc10.3-20.

Problem Xc8.1-4. (Fundamental Matrix)

Find a fundamental matrix $\Phi(t)$ by each of the following methods. Report $e^{At} = \Phi(t)\Phi(0)^{-1}$, using one of the answers for Φ.

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

(a) *Cayley-Hamilton method.* Compute the characteristic equation $\det(A - \lambda I) = 0$. Find two atoms from the roots of this equation. Then $x(t)$ is a linear combination of these atoms. The first equation $x' = 2x - 5y$ can be solved for y to find the second answer.

(b) *Eigenanalysis method.* Find the eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$ of A. Let Φ have columns $e^{\lambda_1 t}\mathbf{v}_1$, $e^{\lambda_2 t}\mathbf{v}_2$.

Problem Xc8.1-12. (Putzer's Method)

The exponential matrix e^{At} can be found in the 2×2 case from Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting $\lambda_2 \to \lambda_1$ [λ_1, t fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute from Putzer's formula e^{At} for the following cases.

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Answer $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc8.1-38. (Laplace's Method)

The exponential matrix e^{At} can be found from the Laplace resolvent formula for the problem $\Phi' = A\Phi$, $\Phi(0) = I$:

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1}\Phi(0) = (sI - A)^{-1}.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$, which implies $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Compute $\Phi(t) = e^{At}$ using the resolvent formula for the following cases.

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc8.2-4. (Variation of Parameters)

Use the variation of parameters formula $\mathbf{u}_p(t) = e^{At} \int e^{-At}\mathbf{f}(t)dt$ to find a particular solution of the given system. Please use technology to do the indicated integration, following the example below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

Example: Solve for $\mathbf{u}_p(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

Problem Xc8.2-19. (Initial Value Problem)

Solve the given initial value problem using a computer algebra system. Follow the example given below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example: Solve for $\mathbf{u}(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The answer is $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$.

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with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t):
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
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End of extra credit problems chapter 8.