Basic Laplace Theory

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The integral
\[ \int_{0}^{\infty} g(t)e^{-st}dt \]
is called the \textbf{Laplace integral} of the function \( g(t) \). It is defined by
\[ \int_{0}^{\infty} g(t)e^{-st}dt \equiv \lim_{N \to \infty} \int_{0}^{N} g(t)e^{-st}dt \]
and it depends on variable \( s \). The ideas will be illustrated for \( g(t) = 1 \), \( g(t) = t \) and \( g(t) = t^2 \). Results appear in Table 1 \textit{infra}.
Laplace Integral or Direct Laplace Transform

The Laplace integral or the direct Laplace transform of a function \( f(t) \) defined for \( 0 \leq t < \infty \) is the ordinary calculus integration problem

\[
\int_{0}^{\infty} f(t)e^{-st} \, dt.
\]

The Laplace integrator is \( dx = e^{-st} \, dt \) instead of the usual \( dt \).

A Laplace integral is succinctly denoted in science and engineering literature by the symbol \( L(f(t)) \), which abbreviates

\[
\int_{E} (f(t)) \, dx,
\]

with set \( E = [0, \infty) \) and Laplace integrator \( dx = e^{-st} \, dt \).
A First Laplace Table

\[ \int_0^\infty (1)e^{-st} \, dt = -\left(\frac{1}{s}\right)e^{-st}\bigg|_{t=0}^{t=\infty} = \frac{1}{s} \]

Laplace integral of \( g(t) = 1 \).

Assumed \( s > 0 \).

\[ \int_0^\infty (t)e^{-st} \, dt = \int_0^\infty -\frac{d}{ds}(e^{-st}) \, dt = -\frac{d}{ds} \int_0^\infty (1)e^{-st} \, dt = -\frac{d}{ds} \left(\frac{1}{s}\right) = \frac{1}{s^2} \]

Laplace integral of \( g(t) = t \).

Use \( L(1) = 1/s \).

Differentiate.

\[ \int_0^\infty (t^2)e^{-st} \, dt = \int_0^\infty -\frac{d}{ds}(te^{-st}) \, dt = -\frac{d}{ds} \int_0^\infty (t)e^{-st} \, dt = -\frac{d}{ds} \left(\frac{1}{s^2}\right) = \frac{2}{s^3} \]

Laplace integral of \( g(t) = t^2 \).

Use \( L(t) = 1/s^2 \).
Summary

Table 1. Laplace integral \( \int_{0}^{\infty} g(t)e^{-st} dt \) for \( g(t) = 1, t \) and \( t^2 \).

\[
\begin{align*}
\int_{0}^{\infty} (1)e^{-st} dt &= \frac{1}{s}, & \int_{0}^{\infty} (t)e^{-st} dt &= \frac{1}{s^2}, & \int_{0}^{\infty} (t^2)e^{-st} dt &= \frac{2}{s^3}.
\end{align*}
\]

In summary, \( L(t^n) = \frac{n!}{s^{1+n}} \)
A Minimal Laplace Table

Solving differential equations by Laplace methods requires keeping a smallest table of Laplace integrals available, usually memorized. The last three entries will be verified later.

Table 2. A minimal Laplace integral table with $L$-notation

<table>
<thead>
<tr>
<th>Integral</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^\infty (t^n)e^{-st} dt$</td>
<td>$L(t^n) = \frac{n!}{s^{1+n}}$</td>
</tr>
<tr>
<td>$\int_0^\infty (e^{at})e^{-st} dt$</td>
<td>$L(e^{at}) = \frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\int_0^\infty (\cos bt)e^{-st} dt$</td>
<td>$L(\cos bt) = \frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$\int_0^\infty (\sin bt)e^{-st} dt$</td>
<td>$L(\sin bt) = \frac{b}{s^2 + b^2}$</td>
</tr>
</tbody>
</table>
The forward table finds the Laplace integral $L(f(t))$ when $f(t)$ is a linear combination of Euler solution atoms. Laplace calculus rules apply to find the Laplace integral of $f(t)$ when it is not in this short table.

### Table 3. Forward Laplace integral table

<table>
<thead>
<tr>
<th>Function $f(t)$</th>
<th>Laplace Integral $L(f(t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{1+n}}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s - a}$</td>
</tr>
<tr>
<td>$\cos bt$</td>
<td>$\frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$\sin bt$</td>
<td>$\frac{b}{s^2 + b^2}$</td>
</tr>
</tbody>
</table>
The backward table finds $f(t)$ from a Laplace integral $L(f(t))$ expression. Always, $f(t)$ is a linear combinations of Euler solution atoms. The Laplace calculus rules apply to find $f(t)$ when it is does not appear in this short table.

### Table 4. Backward Laplace integral table

<table>
<thead>
<tr>
<th>Laplace Integral $L(f(t))$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{s}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\frac{1}{s^{1+n}}$</td>
<td>$t^n/\binom{n}{n!}$</td>
</tr>
<tr>
<td>$\frac{1}{s-a}$</td>
<td>$e^{at}$</td>
</tr>
<tr>
<td>$\frac{s}{s^2+b^2}$</td>
<td>$\cos bt$</td>
</tr>
<tr>
<td>$\frac{1}{s^2+b^2}$</td>
<td>$\frac{\sin bt}{b}$</td>
</tr>
</tbody>
</table>
Some Transform Rules

\[ L(f(t) + g(t)) = L(f(t)) + L(g(t)) \]

The integral of a sum is the sum of the integrals.

\[ L(cf(t)) = cL(f(t)) \]

Constants \( c \) pass through the integral sign.

\[ L(y'(t)) = sL(y(t)) - y(0) \]

The \( t \)-derivative rule, or integration by parts.
Lerch’s Cancelation Law and the Fundamental Theorem of Calculus

\[ L(y(t)) = L(f(t)) \text{ implies } y(t) = f(t) \quad \text{Lerch’s cancelation law.} \]

Lerch’s cancelation law in integral form is

(1) \[ \int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \quad \text{implies } y(t) = f(t). \]

Quadrature Methods

Lerch’s Theorem is used last in Laplace’s quadrature method. In Newton calculus, the quadrature method uses the Fundamental Theorem of Calculus first. The two theorems have a similar use, to isolate the solution \( y \) of the differential equation.
Laplace’s method will be applied to solve the initial value problem

$$y' = -1, \quad y(0) = 0.$$
Table 5. Laplace method details for $y' = -1, y(0) = 0$.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y'(t)e^{-st}dt = -e^{-st}dt$</td>
<td>Multiply $y' = -1$ by $e^{-st}dt$.</td>
</tr>
<tr>
<td>$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$</td>
<td>Integrate $t = 0$ to $t = \infty$.</td>
</tr>
<tr>
<td>$\int_0^\infty y'(t)e^{-st}dt = -1/s$</td>
<td>Use Table 1.</td>
</tr>
<tr>
<td>$s\int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$</td>
<td>Integrate by parts on the left.</td>
</tr>
<tr>
<td>$\int_0^\infty y(t)e^{-st}dt = -1/s^2$</td>
<td>Use $y(0) = 0$ and divide.</td>
</tr>
<tr>
<td>$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$</td>
<td>Use Table 1.</td>
</tr>
<tr>
<td>$y(t) = -t$</td>
<td>Apply Lerch’s cancellation law.</td>
</tr>
</tbody>
</table>
Translation to $L$-notation

Table 6. Laplace method $L$-notation details for $y' = -1$, $y(0) = 0$ translated from Table 5.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(y'(t)) = L(-1)$</td>
<td>Apply $L$ across $y' = -1$, or multiply $y' = -1$ by $e^{-st}dt$, integrate $t = 0$ to $t = \infty$.</td>
</tr>
<tr>
<td>$L(y'(t)) = -1/s$</td>
<td>Use Table 1 forwards.</td>
</tr>
<tr>
<td>$sL(y(t)) - y(0) = -1/s$</td>
<td>Integrate by parts on the left.</td>
</tr>
<tr>
<td>$L(y(t)) = -1/s^2$</td>
<td>Use $y(0) = 0$ and divide.</td>
</tr>
<tr>
<td>$L(y(t)) = L(-t)$</td>
<td>Apply Table 1 backwards.</td>
</tr>
<tr>
<td>$y(t) = -t$</td>
<td>Invoke Lerch’s cancelation law.</td>
</tr>
</tbody>
</table>
1 Example (Laplace method) Solve by Laplace’s method the initial value problem $y' = 5 - 2t$, $y(0) = 1$ to obtain $y(t) = 1 + 5t - t^2$.

**Solution**: Laplace’s method is outlined in Tables 5 and 6. The $L$-notation of Table 6 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$L(y'(t)) = L(5 - 2t)$

Apply $L$ across $y' = 5 - 2t$.

$= 5L(1) - 2L(t)$

Linearity of the transform.

$= \frac{5}{s} - \frac{2}{s^2}$

Use Table 1 forwards.

$sL(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2}$

Apply the $t$-derivative rule.

$L(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}$

Use $y(0) = 1$ and divide.

$L(y(t)) = L(1) + 5L(t) - L(t^2)$

Use Table 1 backwards.

$L(1 + 5t - t^2)$

Linearity of the transform.

$y(t) = 1 + 5t - t^2$

Invoke Lerch’s cancelation law.
2 Example (Laplace method) Solve by Laplace's method the initial value problem
\[ y'' = 10, \ y(0) = y'(0) = 0 \] to obtain \[ y(t) = 5t^2. \]

Solution: The \( L \)-notation of Table 6 will be used to find the solution \( y(t) = 5t^2. \)

\[
L(y''(t)) = L(10) \\
sL(y'(t)) - y'(0) = L(10) \\
s[sL(y(t)) - y(0)] - y'(0) = L(10) \\
s^2L(y(t)) = 10L(1) \\
L(y(t)) = \frac{10}{s^3} \\
L(y(t)) = L(5t^2) \\
y(t) = 5t^2
\]

Apply \( L \) across \( y'' = 10 \).

Apply the \( t \)-derivative rule to \( y' \).

Repeat the \( t \)-derivative rule, on \( y \).

Use \( y(0) = y'(0) = 0 \).

Use Table 1 forwards. Then divide.

Use Table 1 backwards.

Invoke Lerch's cancelation law.