Math 2250 Lab 9

Due Date: 3/26/2015 Name: ______________________

1. **Floating Bodies and the Harmonic Oscillator**.

Archimedes' principle states that a body partially or totally submerged in a liquid is buoyed up by a force equal to the weight of the liquid displaced. The fact that the body is resting or floating on water implies that the downward force due to the weight $W = mg$ of the body must be the same as the upward buoyant force of the water, namely the weight of the water displaced. Call $y = 0$ the equilibrium position of the lower end of a body when it is floating in a liquid. If the body is now depressed a distance y from its equilibrium position, an additional upward force will act on the body due to the additional liquid displaced. If the body's cross-sectional area is A, then the volume of additional liquid displaced is $(Ay)\rho$, where ρ is the weight per unit volume of liquid. Because this additional upward force is the net force acting on the depressed body, then Newton's second law of motion *force* $=$ *mass* $\times$ *acceleration* implies

$$m\frac{d^2y}{dt^2} = -\rho A y,$$

where the positive y-direction is **downward**. This equation is the **classical harmonic oscillator** $y''(t) + \omega^2 y(t) = 0$ with $\omega = \sqrt{\rho A/m}$. A floating body therefore undergoes simple harmonic motion when depressed from its equilibrium position.

Problem 1. Suppose a body of mass m has a cross-sectional area A sq ft. The body is pushed downward y_0 ft from its equilibrium position in a liquid whose weight per ft^3 is ρ and released.

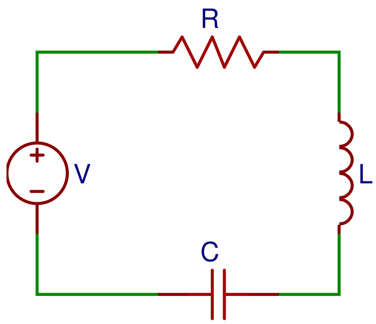
(a) Show details for the following result: The model is

$$\begin{cases} m\dfrac{d^2y}{dt^2} + \rho A y = 0, \\ y(0) = y_0, \quad y'(0) = 0, \end{cases} \tag{1}$$

with unique solution $y(t) = y_0 \cos\left(\sqrt{\dfrac{\rho A}{m}}\, t\right)$.

(b) Assume the body is a right circular cylinder with vertical axis along the y-axis and radius r. Show that the period is $T = \dfrac{2}{r}\sqrt{\dfrac{m\pi}{\rho}}$.

2. **The LC-Circuit Equation and the Method of Undetermined Coefficients**.



Consider the series RLC circuit in the figure, in which the resistor R has been removed ($R = 0$ assumed).

Assumed are zero initial charge $Q(0) = 0$ and zero initial current $Q'(0) = 0$.

Suppose a periodic voltage source $V_0 \cos(\omega t)$ is applied to the circuit. Using Kirchoff's laws, the following initial value problem gives the charge $Q(t)$ on the capacitor.

$$\begin{cases} L \cdot Q''(t) + R \cdot Q'(t) + \dfrac{1}{C} \cdot Q(t) = V_0 \cos(\omega t), \\ Q(0) = 0, \quad Q'(0) = 0. \end{cases}$$

Take $L = 1 \text{ V} \cdot \text{s} \cdot \text{A}^{-1}$, $R = 0\ \Omega$, $C^{-1} = 0.25 \text{ V} \cdot \text{C}^{-1}$, and $V_0 = 4$ V. The left sides of these equations define symbols L, R, C, V_0 using physical units V=volts, s=seconds, A=amperes, Ω=ohms, C=coulombs.

Angular frequency is the coefficient ω_0 in trigonometry terms such as $\cos(\omega_0 t)$, $\cos(\omega_0 t - \alpha)$, $\sin(\omega_0 t)$, $\sin(\omega_0 t - \alpha)$. Angular frequency (natural frequency) may differ from references to *frequency* used in physics and engineering courses.

(a) Find the angular frequency ω_0 of this system, by determing the angular frequency ω_0 of solutions to the unforced equation

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = 0.$$

(b) Assume that $\omega \neq \omega_0$. Let symbol $Q_H(t)$ be the solution to the homogeneous equation, containing arbitrary constants c_1, c_2. Use the *method of undetermined coefficients* to solve for a particular solution $Q_P(t)$. Then use $Q(t) = Q_P(t) + Q_H(t)$ to solve the initial value problem for $Q(t)$ (evaluate c_1, c_2 using $Q(0) = Q'(0) = 0$). Check the answer $Q(t)$ with technology.

(c) Write down the solution $Q(t)$ found in part b) for $\omega = 0.6$, which is a superposition of two cosine functions. Compute the period of this solution. Use technology to graph $Q(t)$ for one period.

(d) Now let $\omega = \omega_0$. Use the *method of undetermined coefficients* to solve for a new particular solution $Q_P(t)$. Then use $Q(t) = Q_P(t) + Q_H(t)$ to solve the initial value problem for $Q(t)$.

(e) The phenomenon of **Beats** and the phenomenon of **Pure Resonance** appear in the solutions obtained above. Explain fully.