Math 2250 Lab 8   Name: _____

Due Date: 03/12/2015, Extra Credits Due Date: 03/26/2015

1. (100 points) The ideas of independence and dependence are explored in parts (a), (b), (c) below for the case of three functions. The tools used are *The Sampling Test, The Wronskian Test*, and the *Definition of Independence*. The domain of all functions will be $-\infty < x < \infty$.

(a) The functions $f_1(x) = 1$, $f_2(x) = \cos(2x)$ and $f_3(x) = \sin^2(x)$ are linearly dependent. The sampling test and the Wronskian test cannot decide dependence. However, attempts to apply either test will give equations for c_1, c_2, c_3 which when solved may help to guess the correct relation

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0, \quad \text{for all } x.$$

The catch is that the relation must hold **for all** x.

The constants c_1, c_2, c_3 are to be found by solving a system of linear homogeneous algebraic equations which has infinitely many solutions (there is a free variable). Try to find the constants using the Wronskian test with $x = 0$. Then justify why the relation above holds for all values of x (not just $x = 0$). In this effort, look for a trig identity.

(b) The functions $f_1(x) = 1$, $f_2(x) = \sin(x)$ and $f_3(x) = \sin^2(x)$ are linearly independent. Display details for both the sampling test and the Wronskian test to confirm independence.

(c) The functions $f_1(x) = 1$, $f_2(x) = \cos(x)$ and $f_3(x) = \cos^2(x)$ are linearly independent. Justify independence by finding three independent Euler solution atoms belonging to the span of the three functions. Cite the theorems used by name or section number.

Used in the details is the double angle trig identity $\cos(2x) = \frac{1}{2} + \frac{1}{2}\cos^2(x)$. Also used is the theorem which says that any finite set of Euler solution atoms is independent.

2. (30 points) **(Extra Credits)** Linear independence and solution spaces

   (a) Given a matrix $\mathbf{A}$, suppose we have a pair of nonzero vectors $\mathbf{v}, \mathbf{w}$ such that $\mathbf{Av} = 0$ and $\mathbf{Aw} \neq 0$

   Show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. (Hint: We can apply $\mathbf{A}$ to the vector $a\mathbf{v} + b\mathbf{w}$, and $\mathbf{A}0 = 0$)

   (b) Consider the linear differential equation

   $$\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

   Show that if $f(x) \neq 0$ solves the differential equation, and $g(x) \neq 0$ does not, then f and g are linearly independent.

3. (30 points) (**Extra Credits**) Consider the differential equation:

$$x^2 y'' + xy' - \frac{1}{2}y = 0.$$

(a) Confirm that this differential equation is linear

(b) The differential equation has a singularity at $x = 0$, so we restrict our attention to the region $x > 0$. On this region, we can use the substitution $x = e^u$ to transform the equation to the constant-coefficient equation of the form

$$\frac{d^2 y}{du^2} + a\frac{dy}{du} + by = 0.$$

Find the resulting differential equation

(c) Find a basis for the solution space of the differential equation you found in part (b)

(d) Rewrite your solutions $y(u)$ in terms of x and verify that they are linearly independent, and satisfy the original equation

4. (40 points) **(Extra Credits)** If a second order linear homogeneous equation with constant coefficients has the following two solutions respectively, find the corresponding differential equation.

(a) $e^{4x} + 3xe^{4x}, (4 - x)e^{4x}$;

(b) $\sin(2x + \frac{\pi}{4}), \sin(2x - \frac{\pi}{4})$