1. Consider the three vectors

\[ \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix} \]

(a) Use a reduced row echelon computation to check that the span of these three vectors is not \( \mathbb{R}^3 \).

(b) Use your reduced row echelon form computation to write \( \mathbf{w} \) as a linear combination of \( \mathbf{u}, \mathbf{v} \). (Hint: what augmented matrix would you have if you were solving \( c_1 \mathbf{u} + c_2 \mathbf{v} = \mathbf{w} \) for \( c_1, c_2 \)?)

(c) The span of these three vectors is actually a plane through the origin. Every plane through the origin can be defined by an implicit equation \( ax + by + cz = 0 \) with some constants \( a, b, c \). Find the constants \( a, b, c \) so that all points \( (x, y, z) \) whose position vectors \( (x, y, z)^T \) are in the span of \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) lie in the plane given by \( ax + by + cz = 0 \).

References: Edwards-Penney Sections 3.3, 4.1.
2. Consider the matrix

\[ A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 4 \\ 0 & 4 & 4 \end{pmatrix} \]

(a) Find a basis for the solution space of

\[ Ax = 0 \]

and determine the dimension of the space.

(b) Find all vectors \( b \in \mathbb{R}^3 \) such that the matrix equation

\[ Ax = b \]

has a solution \( x \). What is a basis for the space of such vectors \( b \)? What is the dimension of this space?

(c) Show that the set of basis vectors from both (a) and (b) constitute a basis for \( \mathbb{R}^3 \).

References: Edwards-Penney Sections 3.3 3.6, 4.1.
3. (a) Suppose we have a matrix \( A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \) and a square determined by points \((0,0), (0,1), (1,1), (1,0)\). What will the image of the square look like under transformation by the matrix \( A \)? i.e.

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

where \((x_1, x_2)^T\) is a point on the original square and \((y_1, y_2)^T\) is its image after transformation by \( A \). Draw the image of the square after this transformation.

(b) Find the area of the parallelogram in (a). (Hint: think of the parallelogram as sitting inside a larger rectangle \(0 \leq y_1 \leq 4, 0 \leq y_2 \leq 4\)). Compare this area to the determinant of \( A \).

(c) Show, for the general \( \mathbf{u}, \mathbf{v} \) in the first quadrant, with \( \mathbf{v} \) counterclockwise from \( \mathbf{u} \), the area of the parallelogram having the vectors \( \mathbf{u}, \mathbf{v} \) as adjacent sides (as in the special case above) always equals \( \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \).

(d) What happens if you compute \( \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} \) instead?

References: Edwards-Penney Sections 3.6, 4.1.