Math 2250 Lab 13 Name: ______________

Due Date: 04/30/2015 Submit Lab 13 under the door 113 JWB.

Problem 1 required. Problems 2 & 3 are extra credit.

1. (100 points) **Predator Prey System**
Consider a predator-prey population consisting of foxes and rabbits living in a certain forest. Let F_k denote the population count of foxes and R_k the population count of rabbits at a given time k. An unlimited food supply is assumed for the rabbits. If there were no foxes, the population of rabbits would grow at a natural rate. Similarly, if there were no rabbits, the fox population would decay, due to no food source.
When both foxes and rabbits are present, we expect the fox-rabbit interactions to inhibit the rabbit population and increase the fox population. This gives the discrete equations

$$\begin{aligned} R_{k+1} &= aR_k - rF_k \\ F_{k+1} &= bF_k + sR_k \end{aligned}$$

where a, b, r and s are positive constants.

System Parameters. Research observations and measurements of the fox and rabbit populations for an interval of time produced the estimates $a = 1.3, b = 0.4, r = 1, q = 0.2$.

   (a) Explain the biological meaning of the constants a, b, r and s.
   Reference: Section 6.3 & 9.3.

   (b) Extinction of both rabbits and foxes means $\lim_{k\to\infty} R_k = \lim_{k\to\infty} F_k = 0$. Without finding the expressions R_k and F_k, explain why both the foxes and the rabbits will die out.

   (c) Find expressions for R_k and F_k in terms of k and initial population R_0 and F_0.

2. (Extra Credit 50 points) **Cell Dynamics**
Consider the cell equations

$$\frac{dp}{dt} = k(H - A - p)$$
$$\frac{dA}{dt} = \epsilon(H - A)$$

where variable p represents the response of the cell and variable A describes the internal state of the cell. Symbols k, H, ϵ are constants. Symbol H represents an external condition driving the response.

(a) Display the nullcline equations. These are the curves obtained from the differential equation by formally setting $dp/dt = 0$ and $dA/dt = 0$.

(b) Find the equilibria for values $H = 1$, $\epsilon = 0.1$, $k = 0.5$.

(c) Draw the phase portrait, including a direction field. Computer assist expected. Assume $H = 1$, $\epsilon = 0.1$, $k = 0.5$.

(d) The equation for A is the same as Newton's law of cooling. Show all linear integrating factor or variables separable details used to obtain the solution

$$A(t) = H + (A_0 - H)e^{-\epsilon t}, \quad A_0 = A(0) = \text{ initial internal state.}$$

(e) Substitute the above solution into the equation for p, to create a new first order linear differential equation. Introduce a new variable $q(t) = e^{kt}p(t)$. Please show all details which are used to arrive at

$$\frac{dq}{dt} = k(H - A_0)e^{(k-\epsilon)t}.$$

(f) Solve for $q(t)$ and report the solution for $p(t)$ with arbitrary initial condition $p(0) = p_0$.

(g) Assume $H = 1$, $\epsilon = 0.1$, $k = 0.5$. Graph the solution $(p(t), A(t))$ as a phase-plane trajectory for several different initial conditions (p_0, A_0) (you are to invent values for p_0, A_0). Are the results consistent with the phase portrait and direction field done earlier?

(h) Compute the limit of the solution $(p(t), A(t))$ at $t = \infty$.

3. (Extra Credit 50 points) **Damped Oscillator as a Dynamical System**
Consider the following initial value problem for a damped mechanical oscillation:

$$x'' + 4x' + 5x = 0.$$

$$x(0) = 2.$$

$$x'(0) = -1.$$

We have learned how to solve the above problem in Chapter 5 by considering the characteristic polynomial of $x'' + 4x' + 5x = 0$. Investigated here is a dynamical systems solution to the damped oscillation problem (Chapter 7 method).

(a) Set $v(t) = x'(t)$. Show all details for transforming the DE $x'' + 4x' + 5x = 0$ into a system of first order equations for variables x and v. The expected answer is

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix},$$

$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b) Solve the dynamical system in part (a) by using the Cayley-Hamilton-Ziebur shortcut method. Reference: Section 7.1, Examples 5, 6, 7. The eigenvalues are complex numbers. The method avoids find eigenvectors and further computations with complex numbers.

(c) The computation finds the solution $x(t)$ for the damped oscillator $x''+4x'+5x = 0$. Find it another way, using Chapter 5. This is an answer check for all preceding effort.