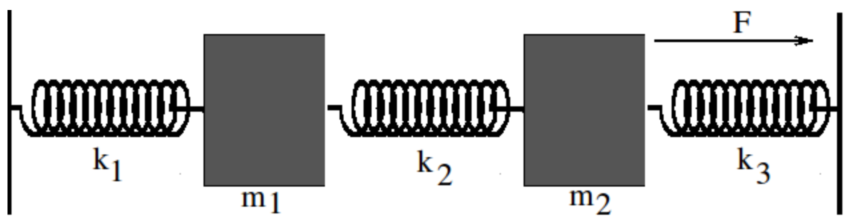
Math 2250 Lab 12 Name: ______________________________

Due Date: 04/09/2015

1. **Using Laplace Transforms to Solve a Linear Second Order System**
Consider a coupled mass-and-spring system as depicted in the figure below and described by the system

$$\begin{cases} 4x'' + 16x - 4y &= 0, \\ 2y'' - \ \ 4x + 6y &= 12\sin 2t \end{cases}$$



A Coupled Spring-Mass System
Symbols $k_1, k_2.k_3$ are Hooke's constants. Symbols m_1, m_2 are masses.
Symbol F is a force acting on mass m_2 with magnitude $f(t) = 12\sin 2t$.
Assumed are values $m_1 = 4, m_2 = 2,\ k_1 = 12, k_2 = 4, k_3 = 2$.

The **purpose of this project** is to solve the system for $x(t), y(t)$ when at $t = 0$ both masses are at rest and in the equilibrium position. The external force $f(t) = 12\sin 2t$ is applied to the second mass m_2 starting at time $t = 0$.

(a) Define $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$. Display the **Laplace Method** details which transform the differential equations and initial conditions ($x(0) = y(0) = x'(0) = y'(0) = 0$) into the 2×2 system of linear algebraic equations

$$(s^2 + 4)X(s) - Y(s) = 0$$

$$(s^2 + 3)Y(s) - 2X(s) = \frac{12}{s^2 + 4}$$

.

(b) Solve the transformed system of part (a) for Laplace transforms $X(s)$ and $Y(s)$. You can use Cramer's Rule for 2×2 systems of linear algebraic equations.

(c) Find the solution $x(t), y(t)$ by backward table methods and partial fractions.

2. The unit staircase function is defined as follows:

$$f(t) = n, n-1 \leq t < n, n = 1, 2, 3, \ldots$$

(a) Sketch the graph of f to see why its name is appropriate.

(b) The function $f(t)$ can also be written as $f(t) = \sum_{n=0}^{\infty} u(t-n)$ where $u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$

Find the Laplace Transform of $f(t)$ by applying the Laplace transform termwise.

(c) Apply the geometric series to obtain the result $\mathcal{L}\{f(t)\} = \dfrac{1}{s(1-e^{-s})}$

3. Consider an RLC Circuit with voltage source $E_0 = 45$ controlled by a switch. Suppose the voltage source is initially turned off. That is, at $t = 0$, $Q(0) = I(0) = 0$. At $t = 30$ seconds, the switch is opened and left open. Let $R = 35\ \Omega$, $L = 0.5$ H, $C = 0.002$ F. This circuit can be modeled as:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) = \begin{cases} 45 & t \geq 30, \\ 0 & t < 30. \end{cases}$$
$$Q(0) = Q'(0) = 0.$$

(a) The Dirac impulse $\delta(t-a)$ is known to satisfy for all piecewise continuous functions f of exponential order the identity

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

Use the identity with $f(t) = e^{-st}$ to find $\mathcal{L}\{\delta(t-a)\}$.

(b) Find the current equation for the circuit by formal differentiation of the charge equation above. It will contain the formal derivative of a unit step, which is a unit Dirac impulse. Then determine the initial conditions $I(0), I'(0)$.

(c) Find the resulting current I(t) in the circuit using Laplace's method.