Math 2250 Lab 12 (Sample) Name: ____________________

1. Solve the system \( x' = x + y, \ y' = x - y + 2, \ x(0) = 0, \ y(0) = 0 \) by Laplace’s Method.

**Solution:** The transformed system is

\[
\begin{align*}
(s - 1)\mathcal{L}(x) + (-1)\mathcal{L}(y) &= 0, \\
(-1)\mathcal{L}(x) + (s + 1)\mathcal{L}(y) &= \mathcal{L}(2).
\end{align*}
\]

Then \( \mathcal{L}(2) = 2/s \) and Cramer’s Rule gives the formulas

\[
\mathcal{L}(x) = \frac{2}{s(s^2 - 2)}, \quad \mathcal{L}(y) = \frac{2(s - 1)}{s(s^2 - 2)}.
\]

After partial fractions, we have

\[
\mathcal{L}(x) = \frac{s}{s^2 - 2} - \frac{1}{s}, \quad \mathcal{L}(y) = \frac{2}{s^2 - 2} - \frac{s}{s^2 - 2} + \frac{1}{s},
\]

and the backward table gives

\[
x = \cosh(\sqrt{2}t) - 1, \quad y = \sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t) + 1.
\]
2. find the following series:

\[ \sum_{k=0}^{\infty} (e^{-s})^k, \]

for \( s > 0 \).

**Solution:** Applying the formula for geometric series \( \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \), with \( r = e^{-s} \).

Then we have

\[ \sum_{k=0}^{\infty} (e^{-s})^k = \frac{1}{1 - e^{-s}}. \]

Note that it is legitimate to use the geometric series because \( s > 0 \), and hence \( |e^{-s}| < 1 \).
3. Suppose a certain spring-mass system with no damping satisfies the initial value problem

\[ x''(t) + 9x(t) = f(t) \]
\[ x(0) = 0, x'(0) = 0 \]

where

\[ f(t) = \begin{cases} 
0 & 0 \leq t < 4 \\
(t - 4)/4 & 4 \leq t \leq 8 \\
1 & t > 8 
\end{cases} \]

(a) Find the general form of the solution in the following cases.

**Case** \( t < 4 \).

**Solution:** For \( t < 4 \), \( x(t) = c_1 \cos(3t) + c_2 \sin(3t) \). The initial conditions imply \( x(t) = 0 \).

**Case** \( t > 8 \).

**Solution:** For \( t > 8 \), \( x(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{9} \) where \( \frac{1}{9} \) is a particular solution of the nonhomogeneous equation and the other two terms compose the complementary or homogeneous solution. Thus, this solution is a simple harmonic oscillator about \( x = \frac{1}{9} \).

(b) Show the details required to rewrite \( f(t) \) in terms of the unit step functions \( u(t - 4) \) and \( u(t - 8) \), as the expression

\[ f(t) = \frac{u(t - 4)(t - 4) - u(t - 8)(t - 8)}{4} \]

(c) Solve the initial value problem using Laplace’s method.
Solution:

\[ L\{x''(t)\} + 9L\{x(t)\} = L\left\{ \frac{u(t-4)(t-4) - u(t-8)(t-8)}{4} \right\} \]

\[ \Rightarrow L\{x''(t)\} + 9L\{x(t)\} = L\left\{ \frac{[u(t-4)](t-4)}{4} \right\} - L\left\{ \frac{[u(t-8)](t-8)}{4} \right\} \]

\[ \Rightarrow s^2 L\{x(t)\} - x'(0) - sx(0) + 9L\{x(t)\} = \frac{e^{-4s} - e^{-8s}}{4s^2} \]

\[ \Rightarrow s^2 X(s) + 9X(s) = \frac{e^{-4s} - e^{-8s}}{4s^2} \]

\[ \Rightarrow (s^2 + 9)X(s) = \frac{e^{-4s} - e^{-8s}}{4s^2} \]

\[ \Rightarrow X(s) = \frac{e^{-4s} - e^{-8s}}{(s^2 + 9)(4s^2)} \]

Let \( H(s) = \frac{1}{s^2(s^2+9)} \). The partial fraction expansion of \( H(s) \) is

\[ H(s) = \frac{1/9}{s^2} - \frac{1/9}{s^2 + 9} \]

Thus, the inverse Laplace transform of \( H(s) \) is

\[ h(t) = \frac{1}{9}t - \frac{1}{27}\sin(3t) \]

and the solution of the initial value problem is

\[ x(t) = \frac{u(t-4)h(t-4) - u(t-8)h(t-8)}{4} \]