Math 2250 Lab 12 (Sample) Name: ____________________

1. Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Solution: The transformed system is

$$\begin{array}{rcrcl} (s-1)\mathcal{L}(x) & + & (-1)\mathcal{L}(y) & = & 0, \\ (-1)\mathcal{L}(x) & + & (s+1)\mathcal{L}(y) & = & \mathcal{L}(2). \end{array}$$

Then $\mathcal{L}(2) = 2/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{2}{s(s^2-2)}, \quad \mathcal{L}(y) = \frac{2(s-1)}{s(s^2-2)}.$$

After partial fractions, we have

$$\mathcal{L}(x) = \frac{s}{s^2-2} - \frac{1}{s}, \quad \mathcal{L}(y) = \frac{2}{s^2-2} - \frac{s}{s^2-2} + \frac{1}{s},$$

and the backward table gives

$$x = \cosh(\sqrt{2}t) - 1, \quad y = \sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t) + 1.$$

2. find the following series:

$$\sum_{k=0}^{\infty}(e^{-s})^k,$$

for $s > 0$.

Solution: Applying the formula for geometric series $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, with $r = e^{-s}$. Then we have

$$\sum_{k=0}^{\infty}(e^{-s})^k = \frac{1}{1-e^{-s}}.$$

Note that it is legitimate to use the geometric series because $s > 0$, and hence $|e^{-s}| < 1$.

3. Suppose a certain spring-mass system with no damping satisfies the initial value problem

$$x''(t) + 9x(t) = f(t)$$

$$x(0) = 0, x'(0) = 0$$

where

$$f(t) = \begin{cases} 0 & 0 \leq t < 4 \\ (t-4)/4 & 4 \leq t \leq 8 \\ 1 & t > 8 \end{cases}$$

(a) Find the general form of the solution in the following cases.

Case $t < 4$.

> **Solution:** For $t < 4$, $x(t) = c_1 \cos(3t) + c_2 \sin(3t)$. The initial conditions imply $x(t) = 0$.

Case $t > 8$.

> **Solution:** For $t > 8$, $x(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{9}$ where $\frac{1}{9}$ is a particular solution of the nonhomogeneous equation and the other two terms compose the complementary or homogeneous solution. Thus, this solution is a simple harmonic oscillator about $x = \frac{1}{9}$.

(b) Show the details required to rewrite $f(t)$ in terms of the unit step functions $u(t-4)$ and $u(t-8)$, as the expression

$$f(t) = \frac{u(t-4)(t-4) - u(t-8)(t-8)}{4}$$

(c) Solve the initial value problem using Laplace's method.

Solution:

$$\begin{aligned}
\mathcal{L}\{x''(t)\} + \mathcal{L}\{9x(t)\} &= \mathcal{L}\{\frac{u(t-4)(t-4) - u(t-8)(t-8)}{4}\} \\
\Rightarrow \mathcal{L}\{x''(t)\} + 9\mathcal{L}\{x(t)\} &= \mathcal{L}\{\frac{[u(t-4)](t-4)}{4}\} - \mathcal{L}\{\frac{[u(t-8)](t-8)}{4}\} \\
\Rightarrow s^2\mathcal{L}\{x(t)\} - x'(0) - sx(0) + 9\mathcal{L}\{x(t)\} &= \frac{e^{-4s} - e^{-8s}}{4s^2} \\
\Rightarrow s^2X(s) + 9X(s) &= \frac{e^{-4s} - e^{-8s}}{4s^2} \\
\Rightarrow (s^2+9)X(s) &= \frac{e^{-4s} - e^{-8s}}{4s^2} \\
\Rightarrow X(s) &= \frac{e^{-4s} - e^{-8s}}{(s^2+9)(4s^2)}
\end{aligned}$$

Let $H(s) = \frac{1}{s^2(s^2+9)}$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/9}{s^2} - \frac{1/9}{s^2+9}$$

Thus, the inverse Laplace transform of $H(s)$ is

$$h(t) = \frac{1}{9}t - \frac{1}{27}\sin(3t)$$

and the solution of the initial value problem is

$$x(t) = \frac{u(t-4)h(t-4) - u(t-8)h(t-8)}{4}$$