The Integrating Factor Method
for a
Linear Differential Equation
\[ y' + p(x)y = r(x) \]

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- Example \( xy' + y = x^2 \).
Consider the homogeneous equation

(1) \[ y' + p(x)y = 0 \]

and the non-homogeneous equation

(2) \[ y' + p(x)y = r(x) \]

where \( p \) and \( r \) are continuous in an interval \( J \).

**Theorem 1 (Superposition)**
The general solution of the non-homogeneous equation (2) is given by

\[ y = y_h + y_p \]

where \( y_h \) is the general solution of homogeneous equation (1) and \( y_p \) is a particular solution of non-homogeneous equation (2).
Variation of Parameters

The initial value problem

\[ y' + p(x)y = r(x), \quad y(x_0) = 0, \]

where \( p \) and \( r \) are continuous in an interval containing \( x = x_0 \), has a particular solution

\[ y(x) = e^{-\int_{x_0}^{x} p(s) \, ds} \int_{x_0}^{x} r(t) e^{\int_{x_0}^{t} p(s) \, ds} \, dt. \]

Formula (4) is called variation of parameters, for historical reasons.

The formula determines a particular solution \( y_p \) which can be used in the superposition identity \( y = y_h + y_p \).

While (4) has some appeal, applications use the integrating factor method, which is developed with indefinite integrals for computational efficiency. No one memorizes (4); they remember and study the method.
Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule

\[
\frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x) \, dx}.
\]

(5)

The factor \( W = e^{\int p(x) \, dx} \) in (5) is called an **integrating factor**.

**Details**

Let \( W = e^{\int p(x) \, dx} \). Then \( W' = pW \), by the rule \((e^x)' = e^x\), the chain rule and the fundamental theorem of calculus \((\int p(x) \, dx)' = p(x)\).

Let’s prove \((WY)' / W = Y' + pY\). The derivative product rule implies

\[
(YW)' = Y'W + YW'
\]

\[
= Y'W + YpW
\]

\[
= (Y' + pY)W.
\]

Divide by \( W \). The proof is complete.
# The Integrating Factor Method

<table>
<thead>
<tr>
<th>Standard Form</th>
<th>Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where $p$, $r$ are continuous. The method applies only in case this is possible.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find $W$</td>
<td>Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.</td>
</tr>
<tr>
<td>Prepare for Quadrature</td>
<td>Obtain the new equation $\frac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (5).</td>
</tr>
<tr>
<td>Method of Quadrature</td>
<td>Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by $W$ to isolate the explicit solution $y(x)$.</td>
</tr>
</tbody>
</table>

Equation (5) is central to the method, because it collapses the two terms $y' + py$ into a single term $(yW)'/W$; the method of quadrature applies to $(yW)' = rW$. Literature calls the exponential factor $W$ an **integrating factor** and equivalence (5) a **factorization** of $y' + p(x)y$. 
**Integrating Factor Example**

**Example.** Solve the linear differential equation $xy' + y = x^2$.

**Solution:** The standard form of the linear equation is

$$y' + \frac{1}{x}y = x.$$

Let

$$W = e^{\int \frac{1}{x} dx} = x$$

and replace the LHS of the differential equation by $(yW)'/W$ to obtain the quadrature equation

$$(yW)' = xW \text{ equivalent to } (yx)' = x^2.$$

Apply quadrature to this equation, then divide by $W$. The answer is

$$y = \frac{x^2}{3} + \frac{C}{x}.$$