Differential Equations and Linear Algebra
2250-1 7:30am on 29 April 2014

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 1-2, 3, 4, 5, 6, 7, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter division adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.

- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.

- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.

- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.

- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.

- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.

- Records will be posted on **CANVAS**, found from the Registrar's web site link. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Homework, quizzes and labs together count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

Name _____ **29Apr2014**

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Scores
Ch1-2.
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch9.
Ch10.

Ch1 and Ch2. (First Order Differential Equations) Complete all problems.

[20%] **Ch1-Ch2(a)**:

Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{2t}v(t)\right) = 20e^t$, $v(0) = 0$

and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 100$.

[20%] **Ch1-Ch2(b)**:
Apply a test to show that $y' = e^x + y\ln|x|$ is not separable. Supply all details.

[10%] **Ch1-Ch2(c)**:
Find all equilibrium solutions for $(yx + 2y)y' = ((2+x)\sin(x)\cos(x) + x)\left(y^2 + 3y + 2\right)$.

[10%] **Ch1-Ch2(d)**:

Solve the homogeneous equation $\dfrac{dy}{dx} - (\cos x)y = 0$.

[20%] **Ch1-Ch2(e)**:
Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = (2+x)(x^2 - 4)(1 - x^2)^5.$$

Label the equilibrium points, display the signs of dx/dt, and classify each equilibrium point as funnel, spout or node.

[10%] **Ch1-Ch2(f)**:
The logistic equation $\frac{dx}{dt} = 3x(x-5)$ has two equilibrium solutions. Find them and classify as extinction state or carrying capacity.

[10%] **Ch1-Ch2(g)**:
Solve the linear drag model $1000\frac{dv}{dt} = 5000 - 100v$.

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Ch3. (Linear Systems and Matrices) Complete all problems.

[40%] **Ch3(a)**: Incorrect answers lose all credit. **Circle the correct answer**.
True or False: **Ch3(a) Part 1**. [10%]:
Assume given a 3×3 matrix A and invertible 3×3 elementary matrices E_1, E_2, E_3.
Define $U = E_3 E_2 E_1 A$. Then U is upper triangular, but U is not always invertible.
True or False: **Ch3(a) Part 2**. [10%]:
If a 3×3 matrix A has no inverse, then there always exists a nonzero vector $\vec{b}$ such that
the equation $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$.
True or False: **Ch3(a) Part 3**. [10%]:
An invertible $n \times n$ matrix A can be written as the product of elementary matrices.
True or False: **Ch3(a) Part 4**. [10%]:
Given a 3×3 matrix A and a 3×1 vector $\vec{b}$, then the system $A\vec{x} = \vec{b}$ can be solved for $\vec{x}$, provided $\vec{b}$ is
a linear combination of the columns of the transpose A^T.

[20%] **Ch3(b)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

[40%] **Ch3(c)**: Determine which values of k correspond to infinitely many solutions
for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 0 & 4 & k \\ 0 & 2-k & 2-k \\ 1 & 4 & 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1/2 \\ k \end{pmatrix}.$$

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Ch4. (Vector Spaces) Complete all problems.

[30%] **Ch4(a)**: Check the independence tests which apply to prove that vectors $x, x^{7/3}, e^x$ are independent in the vector space of all continuous functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define

$$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies independence of f, g, h.

[20%] **Ch4(b)**: Consider the homogenous system $A\vec{x} = \vec{0}$. The nullity of A equals the number of free variables. Give an example of a matrix A with three pivot columns that has nullity 2.

[30%] **Ch4(c)**: Let V be the vector space of all continuously differentiable vector functions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let S be the set of all vector solutions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the dynamical system

$$\begin{cases} x'(t) &= 2x(t) \\ y'(t) &= 4y(t) \end{cases}$$

Find two independent solutions $\vec{v}_1, \vec{v}_2$ such that $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$. *This calculation proves that S is a subspace of V by Picard's theorem and the Span Theorem, hence S is a vector space.*

[20%] **Ch4(d)**: The 4×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 & 3 \end{pmatrix}$$

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 3x^2 + 10xe^{-x} + 4\cos(2x)$ is a solution.

[20%] **Ch5(b)**: Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$(r^4 - 4r^3)((r - \ln(2))^2 + 4)^2 = 0.$$

[30%] **Ch5(c)**: Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} = 11x^2 + 2x + 3 + 12\cos 2x + 13xe^{2x}$$

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[30%] **Ch6(a)**: Let C be a 2×2 matrix having eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right).$$

Find the matrix C.

[30%] **Ch6(b)**: Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 1 & -4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is **not diagonalizable**. Display the details for computing all the eigenpairs. *Details count 75%.*

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[30%] **Ch7(a)**: Incorrect answers lose all credit. **Circle the correct answer**.
True or False: **Ch7(a) Part 1**. [10%]:
A linear dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ cannot always be solved by Laplace theory methods, because the matrix A may fail to be diagonalizable.
True or False: **Ch7(a) Part 2**. [10%]:
A linear 2×2 dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ has solution $\vec{u}(t)$ on $-\infty < t < \infty$ uniquely determined by the initial state $\vec{u}(0)$. The result requires A to be diagonalizable.
True or False: **Ch7(a) Part 3**. [10%]:
Linear dynamical systems $\frac{d}{dt}\vec{u} = A\vec{u}$ exist with A of size 10×10 which can be solved by the linear integrating factor method.

[30%] **Ch7(b)**: Solve the 3×3 linear dynamical system $\vec{u}' = A\vec{u}$, given matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Use the most efficient method possible.

[40%] **Ch7(c)**: Apply the eigenanalysis method to solve the system $\frac{d}{dt}\vec{u} = A\vec{u}$, when the matrix is defined as

$$A = \begin{pmatrix} -7 & 1 & 1 \\ 1 & -7 & 1 \\ 0 & 0 & -7 \end{pmatrix}$$

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a):**
Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

[30%] **Ch9(b):** Consider the nonlinear dynamical system

$$\begin{array}{rcl} x' & = & x - 2y^2 - y + 32, \\ y' & = & 2x^2 - 2xy. \end{array}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4, 4)$ of the linearized system at this equilibrium point.

[40%] **Ch9(c):** Consider the nonlinear dynamical system

$$\begin{array}{rcl} x' & = & -4x + 4y + 9 - x^2, \\ y' & = & 3x - 3y. \end{array}$$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply a theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. _Details count 75%._

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{113}{s^3}$	$\dfrac{3}{3s+4}$	$\dfrac{-8}{s^2+16}$	$\dfrac{\pi s}{s^2+4}$	$\dfrac{e^{-2s}}{s+1}$

$f(t)$	t	$(t+e^{-2t})e^t$	$e^t\cos(2t)$	$t^3 e^{-t}$	$t\cos t$
$\mathcal{L}(f(t))$					

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for the pulse $f(t) = 100$ on $2 \le t < 3$ and $f(t) = 0$ elsewhere.

[30%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-\pi s}}{s^2+9}$.