

# Differential Equations and Linear Algebra

2250-10 7:15am on 6 May 2015

**Instructions.** The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 1-2, 3, 4, 5, 6, 7, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c),  $\dots$ . Each chapter division adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. Please append scratch work for a chapter immediately following chapter solutions. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will not be returned. This policy is due to government privacy rules, which eliminate the possibility of a box of graded exams outside my office.
- Records will be posted on **CANVAS**, found from the Registrar's web site link. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

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**Final Grade.** The final exam counts as two midterm exams. For example, if exam scores earned were 90 and 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 92 + 89 + 89}{4} = 90.$$

Homework, quiz and lab scores are each averages 0–100, weighted respectively 20%, 10%, 10%. The **Coursework Average** =  $\frac{1}{4}(\text{HW} + \text{HW} + \text{Quiz} + \text{Lab})$ . The course average is computed from the formula

$$\text{Course Average} = \frac{60}{100}(\text{Exam Average}) + \frac{40}{100}(\text{Coursework Average}).$$

Averages posted on CANVAS are internal computations of CANVAS: they are not useful numbers for the above formula. You may not earn more than 100% in any category, regardless of extra credit work. Exam scores are records, unalterable by course work or extra credit.

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## Math 2250-10 Sample Final Exam for 7:15am on 6 May 2015

Scores
Ch1-2.
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch9.
Ch10.

**Ch1 and Ch2. (First Order Differential Equations)** Complete all problems.

C Chapter 1: 1.2-7,8,10; 1.3-15,27; 1.4-15,17,39; 1.5-5,17,23,39;

Chapter 2: 2.1-7,17; 2.2-9,17; 2.3-9,23

[20%] **Ch1-Ch2(a):**

Find the position  $x(t)$  from the velocity model  $\frac{d}{dt}(e^{2t}v(t)) = 20e^t$ ,  $v(0) = 0$   
and the position model  $\frac{dx}{dt} = v(t)$ ,  $x(0) = 100$ .

**Answer:**  $v(t) = 20e^{-t} - 20e^{-2t}$ ,  $x(t) = -20e^{-t} + 10e^{-2t} + 110$

[20%] **Ch1-Ch2(b):**

Apply a test to show that  $y' = e^x + y \ln|x|$  is not separable. Supply all details.

**Answer:** Use the partial derivative test for  $y' = f(x, y)$ , which says the equation is not separable provided  $f_y/f$  depends on  $x$ .

[10%] **Ch1-Ch2(c):**

Find all equilibrium solutions for  $(yx + 2y)y' = ((2 + x)\sin(x)\cos(x) + x)(y^2 + 3y + 2)$ .

**Answer:**  $y = -2, y = -1$

[10%] **Ch1-Ch2(d):**

Solve the homogeneous equation  $\frac{dy}{dx} - (\cos x)y = 0$ .

**Answer:** A constant divided by the integrating factor  $W = e^{\int -\cos(x)dx} = e^{-\sin(x)}$ .

[20%] **Ch1-Ch2(e):**

Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = (2 + x)(x^2 - 4)(1 - x^2)^5.$$

Label the equilibrium points, display the signs of  $dx/dt$ , and classify each equilibrium point as funnel, spout or node.

**Answer:** Equilibria  $x = -2, -1, 1, 2$ . Signs of  $dx/dt$  from left to right are **plus, plus, minus, plus, minus**. Node at  $-2$ , Funnel at  $-1$ , Spout at  $1$ , Funnel at  $2$ .

[10%] **Ch1-Ch2(f):**

The logistic equation  $\frac{dx}{dt} = 3x(x - 5)$  has two equilibrium solutions. Find them and classify as extinction state or carrying capacity.

**Answer:**  $x = 0$  is extinction,  $x = 5$  is the carrying capacity.

[10%] **Ch1-Ch2(g):**

Solve the linear drag model  $1000\frac{dv}{dt} = 5000 - 100v$ .

**Answer:** Superposition applies:  $v = v_h + v_p$  where  $v_h$  solves  $1000v'(t) + 100v(t) = 0$  and  $v_p$  is the equilibrium solution. Then  $v_h = \text{constant divided by the integrating factor } W = e^{t/10}$  and  $v_p = 5000/100$ . Answer  $v = ce^{-t/10} + 50$ .

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**Ch3. (Linear Systems and Matrices)** Complete all problems.

Chapter 3: 3.1-16, 3.2-18, 3.3-18, 3.4-22, 3.5-21, 3.6-17, 3.6-32

[40%] **Ch3(a):** Incorrect answers lose all credit. **Circle the correct answer.**True or False: **Ch3(a) Part 1.** [10%]:Assume given a  $3 \times 3$  matrix  $A$  and invertible  $3 \times 3$  elementary matrices  $E_1, E_2, E_3$ .Define  $U = E_3 E_2 E_1 A$ . Then  $U$  is upper triangular, but  $U$  is not always invertible.True or False: **Ch3(a) Part 2.** [10%]:If a  $3 \times 3$  matrix  $A$  has no inverse, then there always exists a nonzero vector  $\vec{b}$  such that the equation  $A\vec{x} = \vec{b}$  has at least one solution  $\vec{x}$ .True or False: **Ch3(a) Part 3.** [10%]:An invertible  $n \times n$  matrix  $A$  can be written as the product of elementary matrices.True or False: **Ch3(a) Part 4.** [10%]:Given a  $3 \times 3$  matrix  $A$  and a  $3 \times 1$  vector  $\vec{b}$ , then the system  $A\vec{x} = \vec{b}$  can be solved for  $\vec{x}$ , provided  $\vec{b}$  is a linear combination of the columns of the transpose  $A^T$ .**Answer:** False. False. True. False.[20%] **Ch3(b):** Define matrix  $A$  and vector  $\vec{b}$  by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

For the system  $A\vec{x} = \vec{b}$ , find  $x_1, x_2$  by Cramer's Rule, showing **all details** (details count 75%).

$$\begin{aligned} \text{Answer: } x_1 &= \Delta_1/\Delta, x_2 = \Delta_2/\Delta, \Delta = \det \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} = 8, \Delta_1 = \det \begin{pmatrix} -3 & 3 \\ 5 & -4 \end{pmatrix} = -3, \\ \Delta_2 &= \det \begin{pmatrix} -2 & -3 \\ 0 & 5 \end{pmatrix} = -10, x_1 = \frac{-3}{8}, x_2 = \frac{-10}{8} = \frac{-5}{4}. \end{aligned}$$

[40%] **Ch3(c):** Determine which values of  $k$  correspond to infinitely many solutions for the system  $A\vec{x} = \vec{b}$  given by

$$A = \begin{pmatrix} 0 & 4 & k \\ 0 & 2-k & 2-k \\ 1 & 4 & 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1/2 \\ k \end{pmatrix}.$$

**Answer:** (1) There is a unique solution for  $\det(A) = k^2 - 6k + 8 \neq 0$ , which implies  $k \neq 2$  and  $k \neq 4$ . Therefore, the answer for infinitely many solutions, if there is one, must have either  $k = 2$  or  $k = 4$  or both. Elimination methods applied to the augmented matrix  $C = \langle A | \vec{b} \rangle$ ,using two swaps and one combo, give the frame  $\begin{pmatrix} 1 & 4 & 5 & 1 \\ 0 & 4 & k & 1 \\ 0 & 0 & A & B \end{pmatrix}$ , where  $A = (2-k)(4-k)/4$ and  $B = (k-4)/4$ . The no solution case must have a signal equation. The infinite solution case must have a free variable but no signal equation. Then (2) No solution for  $k = 2$  [equation 2 is a signal equation]; (3) Infinitely many solutions for  $k = 4$  [a free variable  $x_3$  but no signal equation].

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**Ch4. (Vector Spaces)** Complete all problems.

Chapter 4: 4.1-16, 4.2-28, 4.3-18, 4.4-20, 4.5-22

[30%] **Ch4(a):** Check the independence tests which apply to prove that vectors  $x, x^{7/3}, e^x$  are independent in the vector space of all continuous functions on  $-\infty < x < \infty$ .

- |                          |                         |   |
|--------------------------|-------------------------|---|
| <input type="checkbox"/> | <b>Wronskian test</b>   | Wronskian of functions $f, g, h$ nonzero at $x = x_0$ implies independence of $f, g, h$ .                           |
| <input type="checkbox"/> | <b>Rank test</b>        | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.                     |
| <input type="checkbox"/> | <b>Determinant test</b> | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant. |
| <input type="checkbox"/> | <b>Atom test</b>        | Any finite set of distinct Euler solution atoms is independent.   |
| <input type="checkbox"/> | <b>Pivot test</b>       | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix $A$ has 3 pivot columns.        |
| <input type="checkbox"/> | <b>Sampling test</b>    | Let samples $a, b, c$ be given and for functions $f, g, h$ define   |

$$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$

Then  $\det(A) \neq 0$  implies independence of  $f, g, h$ .

**Answer:** Tests 2, 3, 5 fail to apply, because these tests are about fixed vectors, not functions. Details for the other tests can be given: let  $f(x) = x, g(x) = x^{7/3}, h(x) = e^x$ . These are not atoms, so the atom test does not apply. The Wronskian test applies directly, using  $x_0 = 1$  to obtain Wronskian determinant value  $W = 4e/3$ . The sampling test applies using samples  $a = 0$ ,

$$b = 1, c = 2 \text{ because then } A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & e \\ 2 & 4\sqrt[3]{2} & e^2 \end{pmatrix} \text{ and } \det(A) \neq 0.$$

[20%] **Ch4(b):** Consider the homogenous system  $A\vec{x} = \vec{0}$ . The nullity of  $A$  equals the number of free variables. Give an example of a matrix  $A$  with three pivot columns that has nullity 2.

**Answer:** Let  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  be any three columns of the identity and let  $\vec{v}_4 = \vec{v}_5$  each be the zero vector. Define  $A$  to be the augmented matrix of these four vectors. Then  $A$  has 5 columns. There are three pivot columns and two free variables  $x_4, x_5$ , hence  $A$  has 3 pivots and nullity 2.

[30%] **Ch4(c):** Let  $V$  be the vector space of all continuously differentiable vector functions  $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ . Let  $S$  be the set of all vector solutions  $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$  of the dynamical system

$$\begin{cases} x'(t) &= 2x(t) \\ y'(t) &= 4y(t) \end{cases}$$

Find two independent solutions  $\vec{v}_1, \vec{v}_2$  such that  $S = \text{span}(\vec{v}_1, \vec{v}_2)$ . This calculation proves that  $S$  is a subspace of  $V$  by Picard's theorem and the Span Theorem, hence  $S$  is a vector space.

**Answer:** The dynamical system is diagonal, therefore it can be solved by the method of linear cascades (Section 1.5). The general solution is  $x = c_1 e^{2t}, y = c_2 e^{4t}$ . Then two independent vector solutions are found by taking partial derivatives on the symbols  $c_1, c_2$ , obtaining  $\vec{v}_1 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$  and  $\vec{v}_2 = \begin{pmatrix} 0 \\ e^{4t} \end{pmatrix}$ . Then any solution of the dynamical system is given by  $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2$ .

[20%] **Ch4(d):** The  $4 \times 6$  matrix  $A$  below has some independent columns. Report the independent columns of  $A$ , according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 & 3 \end{pmatrix}$$

**Answer:** Find  $\text{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ . The pivot columns are 2 and 3.

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**Ch5. (Linear Equations of Higher Order)** Complete all problems.

Chapter 5: 5.1-33 to 5.1-42, 5.3-15, 5.3-28, 5.4-18, 5.5-4,

Chapter 5: 5.5-24, 5.5-48, 5.6-12, 5.6-18

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that  $y = 3x^2 + 10xe^{-x} + 4\cos(2x)$  is a solution.

**Answer:** The atoms  $x^2$ ,  $xe^{-x}$ ,  $\cos(2x)$  correspond to roots  $0, 0, 0, -1, -1, 2i, -2i$ . The factor theorem implies the characteristic polynomial should be  $r^3(r+1)^2(r^2+4)$ .

[20%] **Ch5(b)**: Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$(r^4 - 4r^3)((r - \ln(2))^2 + 4)^2 = 0.$$

**Answer:** The roots are  $0, 0, 0, 4, \ln(2) \pm 2i, \ln(2) \pm 2i$ . By Euler's theorem, a basis is the set of atoms for these roots:  $1, x, x^2, e^{4x}, e^{\ln(2)x} \cos(2x), e^{\ln(2)x} \sin(2x), xe^{\ln(2)x} \cos(2x), xe^{\ln(2)x} \sin(2x)$ . The exponential factor  $e^{\ln(2)x}$  can be written  $2^x$ .

[30%] **Ch5(c)**: Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40 \cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies.

**Answer:** Use undetermined coefficients trial solution  $x = d_1 \cos 4t + d_2 \sin 4t$ . Then  $d_1 = 5/6$ ,  $d_2 = 0$ . Then  $x_p(t) = (5/6) \cos(4t)$ . The characteristic equation  $r^2 + 64 = 0$  has roots  $\pm 8i$  and the corresponding Euler solution atoms are  $\cos(8t), \sin(8t)$ . Then  $x_h(t) = c_1 \cos(8t) + c_2 \sin(8t)$ . The general solution is  $x = x_h + x_p$ . Now use  $x(0) = x'(0) = 0$  to determine  $c_1 = -5/6, c_2 = 0$ . Then  $x(t) = -(5/6) \cos(8t) + (5/6) \cos(4t)$ .

[30%] **Ch5(d)**: Determine the **shortest** trial solution for  $y_p$  according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4 y}{dx^4} - 4 \frac{d^2 y}{dx^2} = 11x^2 + 2x + 3 + 12 \cos 2x + 13xe^{2x}$$

**Answer:** The homogeneous problem has roots  $0, 0, \pm 2i$  with atoms  $1, x, e^{2x}, e^{-2x}$ . The trial solution is constructed initially from  $f(x) = 11x^2 + 2x + 3 + 12 \cos 2x + 13xe^{2x}$ , which has seven atoms in a list in four groups (1)  $1, x, x^2$ ; (2)  $\cos 2x$ ; (3)  $\sin 2x$ ; (4)  $e^{2x}, xe^{2x}$ . Conflicts with the homogeneous equation atoms causes a repair of groups (1), (4) making the new groups (1)  $x^2, x^3, x^4$ ; (2)  $\cos 2x$ ; (3)  $\sin 2x$ ; (4)  $xe^{2x}, x^2e^{2x}$ . Then the shortest trial solution is a linear combination of the seven atoms in the corrected list.

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**Ch6. (Eigenvalues and Eigenvectors)** Complete all problems.

Chapter 6: 6.1-23, 6.1-35, 6.2-25, 6.2-31

[30%] **Ch6(a):** Let  $C$  be a  $2 \times 2$  matrix having eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Find the matrix  $C$ .

**Answer:** Write  $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ . Then  $CP = PD$  implies  $C = PDP^{-1} =$

$$\begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}.$$

[30%] **Ch6(b):** Find the eigenvalues of the matrix  $B$ :

$$B = \begin{pmatrix} 2 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

**Answer:** The characteristic polynomial is  $\det(B - rI) = (2 - r)(5 - r)(4 - r)(2 - r)$ . The eigenvalues are 2, 2, 4, 5.

It is possible to directly find the eigenvalues of  $B$  by cofactor expansion of  $|B - rI|$ . This is the expected method. Expand  $|B - rI|$  by cofactors on row 2. The calculation reduces to a  $3 \times 3$  determinant. Expand the  $3 \times 3$  along row 1 to reduce the calculation to a  $2 \times 2$ , which evaluates as a quadratic by Sarrus' Rule.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix  $B$  is a block matrix  $B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$ , where  $B_1, B_2, B_3$  are all  $2 \times 2$  matrices. Then

$B - rI = \begin{pmatrix} B_1 - rI & B_2 \\ 0 & B_3 - rI \end{pmatrix}$ . Using the determinant product theorem for such special block

matrices (zero in the left lower block) gives  $|B - rI| = |B_1 - rI||B_3 - rI|$ . Because a determinant equals the determinant of its transpose, then the answer is that  $B$  has eigenvalues equal to the eigenvalues of  $B_1$  and  $B_3$ , even if the zero block is in the upper right. The two  $2 \times 2$  determinants are quickly found by Sarrus' Rule:  $|B_1 - rI| = (2 - r)(5 - r)$  and  $|B_3 - rI| = r^2 - 6r + 8 = (4 - r)(2 - r)$ .

[40%] **Ch6(c):** The matrix  $A = \begin{pmatrix} 1 & -4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$  is **not diagonalizable**. Display the details for computing all the eigenpairs. *Details count 75%.*

**Answer:** The characteristic equation is  $(3 - r)^3 = 0$  with roots 3, 3, 3. Because  $A - 3I =$

$$\begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has reduced row echelon form  $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , then there are two eigenpairs

for  $\lambda = 3$ , with eigenvectors  $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,  $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ .

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**Ch7. (Linear Systems of Differential Equations)** Complete all problems.

Chapter 7: 7.1-18, 7.2-16, 7.3-18, 7.3-28

[30%] **Ch7(a):** Incorrect answers lose all credit. **Circle the correct answer.**True or False: **Ch7(a) Part 1.** [10%]:A linear dynamical system  $\frac{d}{dt}\vec{u} = A\vec{u}$  cannot always be solved by Laplace theory methods, because the matrix  $A$  may fail to be diagonalizable.True or False: **Ch7(a) Part 2.** [10%]:A linear  $2 \times 2$  dynamical system  $\frac{d}{dt}\vec{u} = A\vec{u}$  has solution  $\vec{u}(t)$  on  $-\infty < t < \infty$  uniquely determined by the initial state  $\vec{u}(0)$ . The result requires  $A$  to be diagonalizable.True or False: **Ch7(a) Part 3.** [10%]:Linear dynamical systems  $\frac{d}{dt}\vec{u} = A\vec{u}$  exist with  $A$  of size  $10 \times 10$  which can be solved by the linear integrating factor method.**Answer:** False. False. True.[30%] **Ch7(b):** Solve the  $3 \times 3$  linear dynamical system  $\vec{u}' = A\vec{u}$ , given matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Use the most efficient method possible.

**Answer:** The matrix is triangular, so the theory of linear cascades applies. First,  $x'_3 = 2x_3$ , then  $x_3 = c_1 e^{2t}$ . Back-substitute:  $x'_2 = 2x_3 = 2c_1 e^{2t}$  implies  $x_2 = c_1 e^{2t} + c_2$ . Then  $x'_1 = 2x_2 = 2c_1 e^{2t} + 2c_2$ , which implies  $x_1 = c_1 e^{2t} + 2c_2 t + c_3$ . The method produces a vector answer  $\vec{u} = \Phi(t)\vec{c}$ , where  $\vec{c}$  has components  $c_1, c_2, c_3$ . The matrix  $\Phi(t)$  is called a fundamental matrix. Some fundamental matrices, obtained from this method:

$$\begin{pmatrix} e^{2t} & 2t & 1 \\ e^{2t} & 1 & 0 \\ e^{2t} & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & t & e^{2t} \\ 1 & 1/2 & e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

Does Cayley-Hamilton-Ziebur work? Yes, it always works. The roots of  $|A - rI| = 0$  are  $r = 0, 0, 2$ , which implies atoms  $1, t, e^{2t}$  and then  $\vec{u} = \vec{d}_1 + \vec{d}_2 t + \vec{d}_3 e^{2t}$ . Differentiate this relation two times and set  $t = 0$  in the resulting 3 equations in 3 vector unknowns  $\vec{d}_1, \vec{d}_2, \vec{d}_3$ . Solve the system

for the unknown vectors, in terms of  $A$  and  $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \vec{c}$ . You will get  $\vec{d}_3 = \frac{1}{4} A^2 \vec{c}$ ,

$\vec{d}_2 = A\vec{c} - \frac{1}{2} A^2 \vec{c}$ ,  $\vec{d}_1 = (I - \frac{1}{4} A^2) \vec{c}$ . Then  $\vec{u}(t) = \left( I - \frac{1}{4} A^2 + (A - \frac{1}{2} A^2)t + \frac{1}{4} A^2 e^{2t} \right) \vec{c}$ . This answer is slightly different than what was obtained by linear cascades, and it takes more time to produce.

Does the Eigenanalysis method apply? The roots of the characteristic equation are  $0, 0, 2$ , and the corresponding atoms are  $1, t, e^{2t}$ . But  $A - 0I$  is just  $A$ , which has rank 2 and nullity 1. There is only one eigenpair:  $A$  is not diagonalizable. The eigenanalysis method *does not apply*.

Laplace theory? Yes, it always works. Begin with the Laplace resolvent equation  $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) = \vec{c}$ . Solve as  $\mathcal{L}(\vec{u}) = (sI - A)^{-1} \vec{c}$ . Using the backward Laplace table on each entry

in  $(sI - A)^{-1}$  gives the answer  $\vec{u}(t) = \begin{pmatrix} 1 & t & a(t) \\ 0 & 1 & b(t)t \\ 0 & 0 & c(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ , where  $a(t) = -t + (e^{2t} -$

$1)/2, b(t) = e^{2t} - 1, c(t) = e^{2t}$ . None of this is easy, and the preferred method is linear cascades.

[40%] **Ch7(c):** Apply the eigenanalysis method to solve the system  $\frac{d}{dt}\vec{u} = A\vec{u}$ , when the matrix is defined as

$$A = \begin{pmatrix} -7 & 1 & 1 \\ 1 & -7 & 1 \\ 0 & 0 & -7 \end{pmatrix}$$



**Answer:** The eigenpairs of  $A$  are  $(-8, \vec{v}_1), (-7, \vec{v}_2), (-6, \vec{v}_3)$  where

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\vec{u}(t) = c_1 e^{-8t} \vec{v}_1 + c_1 e^{-7t} \vec{v}_2 + c_1 e^{-6t} \vec{v}_3.$$

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**Ch9. (Nonlinear Systems)** Complete all problems.

Chapter 9: 9.1-8, 9.1-18, 9.2-2, 9.2-12, 9.2-22, 9.3-28, 9.4-8

[30%] **Ch9(a):**

Determine whether the unique equilibrium  $\vec{u} = \vec{0}$  is stable or unstable. Then classify the equilibrium point  $\vec{u} = \vec{0}$  as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

**Answer:** It is an unstable spiral. Details: The eigenvalues of  $A$  are roots of  $r^2 + 2r + 5 = (r + 1)^2 + 4 = 0$ , which are complex conjugate roots  $1 \pm 2i$ . Rotation eliminates the saddle and node. Finally, the atoms  $e^t \cos 2t$ ,  $e^t \sin 2t$  have limit zero at  $t = -\infty$ , therefore the system is stable at  $t = -\infty$  and unstable at  $t = \infty$ . So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

[30%] **Ch9(b):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is  $x = 4$ ,  $y = 4$ . Compute the Jacobian matrix  $A = J(4, 4)$  of the linearized system at this equilibrium point.

**Answer:** The Jacobian is  $J(x, y) = \begin{pmatrix} 1 & -4y - 1 \\ 4x - 2y & -2x \end{pmatrix}$ . Then  $A = J(4, 4) = \begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$ .

[40%] **Ch9(c):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -4x + 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned}$$

At equilibrium point  $x = 3$ ,  $y = 3$ , the Jacobian matrix is  $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$ .

- (1) Determine the stability at  $t = \infty$  and the phase portrait classification saddle, center, spiral or node at  $\vec{u} = \vec{0}$  for the linear system  $\frac{d}{dt}\vec{u} = A\vec{u}$ .
- (2) Apply a theorem to classify  $x = 3$ ,  $y = 3$  as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

**Answer:** (1) The Jacobian is  $J(x, y) = \begin{pmatrix} -4 - 2x & 4 \\ 3 & -3 \end{pmatrix}$ . Then  $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$ .

The eigenvalues of  $A$  are found from  $r^2 + 13r + 18 = 0$ , giving distinct real negative roots  $-\frac{13}{2} \pm (\frac{1}{2})\sqrt{97}$ . Because there are no trig functions in the Euler solution atoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at  $t = \infty$ , therefore it is a node and we report a **stable node** for the linear problem  $\vec{u}' = A\vec{u}$  at equilibrium  $\vec{u} = \vec{0}$ . (2) Theorem 2 in Edwards-Penney section 9.2 applies to say that the same is true for the nonlinear system: **stable node** at  $x = 3$ ,  $y = 3$ . The exceptional case in Theorem 2 is a proper node, having characteristic equation roots that are equal. Stability is always preserved for nodes.

## Math 2250-10 Sample Final Exam for 7:15am on 6 May 2015

**Ch10. (Laplace Transform Methods)** Complete all problems.

Chapter 10: 10.1-29, 10.2-6, 10.2-12, 10.2-23, 10.3-8, 10.3-21,

Chapter 10: 10.3-38, 10.4-13, 10.4-18, 10.5-9

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a):** Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\frac{113}{s^3}$	$\frac{3}{3s+4}$	$\frac{-8}{s^2+16}$	$\frac{\pi s}{s^2+4}$	$\frac{e^{-2s}}{s+1}$

$f(t)$	$t$	$(t + e^{-2t})e^t$	$e^t \cos(2t)$	$t^3 e^{-t}$	$t \cos t$
$\mathcal{L}(f(t))$					

**Answer:** First table left to right:  $\frac{113t^2}{2},$

$e^{-4t/3},$

$-2 \sin 4t,$

$\pi \cos 2t,$

$e^{-(t-2)}u(t-2).$

The unit step  $u(t)$  is defined in the Edwards-Penney textbook,  $u(t) = 1$  for  $t \geq 0$ , zero elsewhere.

Second table left to right:  $\frac{1}{s^2},$

$$\frac{1}{(s-1)^2} + \frac{1}{s+1},$$

$$\left. \frac{s}{s^2+4} \right|_{s \rightarrow (s-1)} = \frac{s-1}{(s-1)^2+4},$$

$$\left. \frac{6}{s^4} \right|_{s \rightarrow (s+1)} = \frac{6}{(s+1)^4},$$

$$-\frac{d}{ds} \frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$$

[30%] **Ch10(b):** Compute  $\mathcal{L}(f(t))$  for the pulse  $f(t) = 100$  on  $2 \leq t < 3$  and  $f(t) = 0$  elsewhere.

**Answer:** Define  $u(t)$  to be the unit step. Use  $f(t) = 100(u(t-2) - u(t-3))$  and the second shifting theorem. Then  $\mathcal{L}(100u(t-2)) = e^{-2s}\mathcal{L}(100|_{t \rightarrow t+2}) = e^{-2s}\mathcal{L}(100) = 100e^{-2s}/s$ . Similarly,  $\mathcal{L}(100u(t-3)) = 100e^{-3s}/s$ . The answer:  $\mathcal{L}(f(t)) = 100(e^{-2s} - e^{-3s})/s$ .

[30%] **Ch10(c):** Solve for  $f(t)$  in the equation  $\mathcal{L}(f(t)) = \frac{e^{-\pi s}}{s^2+9}$ .

**Answer:** First write  $\frac{1}{s^2+9} = \mathcal{L}(\frac{1}{3}\sin(3t))$  using the backward Laplace table. Apply the second shifting theorem:  $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)u(t-a))$ . Then  $\mathcal{L}(f(t)) = e^{-\pi s}\mathcal{L}(\frac{1}{3}\sin(3t)) = \mathcal{L}(\frac{1}{3}\sin(3z)u(z))\Big|_{z=t-\pi} = \frac{1}{3}\mathcal{L}(\sin(3t-3\pi)u(t-\pi))$ . Lerch's theorem implies  $f(t) = \frac{1}{3}\sin(3t-3\pi)u(t-\pi) = -\frac{1}{3}\sin(3t)u(t-\pi)$ .