Differential Equations and Linear Algebra

2250-10 7:15am on 6 May 2015

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 1-2, 3, 4, 5, 6, 7, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter division adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. Please append scratch work for a chapter immediately following chapter solutions. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will not be returned. This policy is due to government privacy rules, which eliminate the possibility of a box of graded exams outside my office.
- Records will be posted on **CANVAS**, found from the Registrar's web site link. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90 and 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 92 + 89 + 89}{4} = 90.$$

Homework, quiz and lab scores are each averages 0–100, weighted respectively 20%, 10%, 10%. The **Coursework Average** = $\frac{1}{4}$(HW+HW+Quiz+Lab). The course average is computed from the formula

$$\text{Course Average} = \frac{60}{100}(\text{Exam Average}) + \frac{40}{100}(\text{Coursework Average}).$$

Averages posted on CANVAS are internal computations of CANVAS: they are not useful numbers for the above formula. You may not earn more than 100% in any category, regardless of extra credit work. Exam scores are records, unalterable by course work or extra credit.

Please recycle this page or keep it for your records.

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Scores
Ch1-2.
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch9.
Ch10.

Ch1 and Ch2. (First Order Differential Equations)

[20%] **Ch1-Ch2(a)**:
Find the position $x(t)$ from the velocity model $\frac{d}{dt}\left(e^t v(t)\right) = 10e^{2t}$, $v(0) = 0$ and the position model $\frac{dx}{dt} = v(t)$, $x(0) = 100$.

[10%] **Ch1-Ch2(b)**:
Find all equilibrium solutions for $y' = x^3 e^y (2 + \cos(y))(y^2 - 3y + 2)$.

[20%] **Ch1-Ch2(c)**:
Given $y' = \frac{2x^2 + x}{1 + x}\left(\frac{y^4 - 2y^2 + 1}{y}\right)$,
find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

[20%] **Ch1-Ch2(d)**:
Solve the linear homogeneous equation $2\sqrt{1 + x}\frac{dy}{dx} = y$ using the integrating factor shortcut.

[10%] **Ch1-Ch2(e)**:
Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = (3 + x)(x^2 - 9)(4 - x^2)^3.$$

Label the equilibrium points, display the signs of dx/dt, and classify each equilibrium point as funnel, spout or node. To save time, **do not draw** a phase portrait.

[20%] **Ch1-Ch2(f)**:
Solve the linear drag model $1000\frac{dv}{dt} = 50 - 200v$ using superposition $v = v_h + v_p$.

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Ch3. (Linear Systems and Matrices)

[40%] **Ch3(a)**: Consider a 3×5 matrix A and its reduced row echelon form:

$$B = \mathbf{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(1) Explain in detail why $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have exactly the same solutions.

(2) Show the linear algebra steps used to find the scalar general solution to the system $B\vec{x} = \vec{0}$.

(3) Report a basis for the solution space S of $A\vec{x} = \vec{0}$.

(4) Report the dimension of S.

[20%] **Ch3(b)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

[40%] **Ch3(c)**:

Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 0 & k-2 & k-3 \\ 1 & 4 & k \\ 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ k \end{pmatrix}.$$

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Ch4. (Vector Spaces)

[30%] **Ch4(a)**: Some independence tests below apply to prove that vectors x, x^2, xe^x are independent in the vector space of all continuous functions on $-\infty < x < \infty$. Mark one method and display the details of application (details count 75%).

☐ **Wronskian test** Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.

☐ **Atom test** Any finite set of distinct Euler solution atoms is independent.

☐ **Sampling test** Let samples a, b, c be given and for functions f, g, h define

$$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies independence of f, g, h.

[20%] **Ch4(b)**: Give an example of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for which the nullity of their augmented matrix A is two.

[20%] **Ch4(c)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fraction decomposition of the fraction

$$\frac{3x^2 - 14x + 3}{(x+1)^2(x-2)^2}$$

To save time, **do not solve** for a, b, c, d.

[30%] **Ch4(d)**:

The 5×6 matrix A below has some independent columns. Report a largest set of independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -2 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 6 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

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Ch5. (Linear Equations of Higher Order)

[10%] **Ch5(a)**: Find a basis for the solution space of $y'' + 4y' + 5y = 0$.

[20%] **Ch5(b)**: Solve for the general solution y of the equation $\frac{d^6y}{dx^6} + 16\frac{d^4y}{dx^4} = 0$.

[20%] **Ch5(c)**: Find a basis for the solution space of a linear constant coefficient homogeneous differerential equaiton, given the characteristic equation is $r(r+1)(r^3 - r)^2(r^2 + 2r + 5)^2 = 0$.

[20%] **Ch5(d)**: Given $6x''(t) + 2x'(t) + 2x(t) = 5\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 6$, $c = 2$, $k = 2$, answer the following questions.

True $\boxed{}$ or False $\boxed{}$. Practical mechanical resonance is at input frequency $\omega = \sqrt{5/2}$.

True $\boxed{}$ or False $\boxed{}$. The homogeneous problem is over-damped.

[30%] **Ch5(e)**: Determine for $\frac{d^5y}{dx^5} + 4\frac{d^3y}{dx^3} = x + x^2 + e^x + x\cos(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

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Ch6. (Eigenvalues and Eigenvectors)

[20%] **Ch6(a)**: Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

[20%] **Ch6(b)**: Find the 2×2 matrix A which has eigenpairs

$$\left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

[20%] **Ch6(c)**: Find the eigenvectors corresponding to complex eigenvalues $-1 \pm 3i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

[30%] **Ch6(d)**: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$. Display the details for finding all eigenpairs of A.

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Ch7. (Linear Systems of Differential Equations)

The methods studied are (1) Linear Cascade method, (2) Cayley-Hamilton-Ziebur method and the 2×2 shortcut, (3) Eigenanalysis method, (4) Laplace's method, including the Laplace resolvent shortcut.

[30%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned} \frac{dx}{dt} &= x + y, \\ \frac{dy}{dt} &= 6x + 2y. \end{aligned}$$

[30%] **Ch7(b)**: Consider the scalar system

$$\begin{cases} x' &= 3x \\ y' &= x, \\ z' &= x + y \end{cases}$$

Report two possible methods that apply to solve for x, y, z. Then choose one method and display the solution details and the answer (details count 75%).

[40%] **Ch7(c)**: Define

$$A = \begin{pmatrix} -3 & 4 & -10 \\ 0 & 2 & 0 \\ 5 & -4 & 12 \end{pmatrix}$$

The eigenvalues of A are 7, 2, 2. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\mathbf{u}' = A\mathbf{u}$.

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Ch9. (Nonlinear Systems)

[10%] **Ch9(a)**: Which of the four types *center, spiral, node, saddle* can be asymptotically stable at $t = \infty$? Explain your answer.

In parts (b), (c), (d), (e) below, consider the **nonlinear system**

$$x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy. \tag{1}$$

[20%] **Ch9(b)**: Display details for finding the equilibrium points for the nonlinear system (1). There are four answers, one of which is $(4, 6)$.

[20%] **Ch9(c)**: Consider again system (1). Compute the Jacobian matrix at (x, y). Then compute the Jacobian matrix at equilibrium point $(4, 6)$.

[20%] **Ch9(d)**: Classify the linear system for equilibrium $(4, 6)$ as a node, spiral, center, saddle.

[30%] **Ch9(e)**: Consider again system (1). What classification can be deduced for equilibrium $(4, 6)$ of nonlinear system (1), according to the Pasting Theorem? Explain fully (details count 75%).

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Ch10. (Laplace Transform Methods)

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\frac{2s}{1+s^2}$	$\frac{2}{3s+1}$	$\frac{5}{s^2+4}$	$\frac{2}{s^2}+\frac{1}{s}$	$\frac{1}{(s+1)^2+4}$

$f(t)$	t	$(1+e^{-t})^2$	$\cos(t)$	$e^{-t}\cos(t)$	$t\sin t$
$\mathcal{L}(f(t))$					

[30%] **Ch10(b)**: Let u be the unit step, $u(t)=1$ for $t\ge 0$, $u(t)=0$ for $t<0$. Compute $\mathcal{L}(x(t)$, given the mechanical problem

$$x''(t)+4x(t)=5(u(t-1)-u(t-2)),\quad x(0)=x'(0)=0.$$

To save time, **do not solve** for $x(t)$.

[30%] **Ch10(c)**: Solve for $g(t)$ in the equation $\mathcal{L}(g(t))=\dfrac{e^{-2s}}{s+5}$.