Differential Equations and Linear Algebra 2250

Sample Midterm Exam 2

Exam Date: 17 April 2015 at 7:25am

Instructions: This in-class exam is designed to be completed in 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. This sample contains extra sample problems. The actual exam is certainly much shorter, tested for 80 minutes.

Chapter 4

1. **(Chapter 4)** Do **all parts**.

(a) State a dependence test for 3 vectors in $\mathcal{R}^4$. Write the hypothesis and conclusion, not just the name of the test.

(**(b)** State fully an independence test for 3 polynomials. It should apply to show that 1, $1+x$, $x(1+x)$ are independent.

(c) For any matrix A, $\mathbf{rank}(A)$ equals the number of lead variables for the problem $A\vec{x} = \vec{0}$. How many non-pivot columns in an 8×8 matrix A with $\mathbf{rank}(A) = 6$?

(d) Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & -2 & 0 & -6 & 0 \\ 0 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{pmatrix}.$$

Decide if the four rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are independent and display the details of the chosen independence test.

(e) Extract from the list below a largest set of independent vectors.

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \vec{v_5} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \vec{v_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

(e) Check the independence tests which apply to prove that vectors $x, x^{7/3}, e^x$ are independent in the vector space of all continuous functions on $-\infty < x < \infty$. Demerits are given for missing a box, and also for checking a box that does not apply.

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define $$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$ Then $\det(A) \neq 0$ implies independence of f, g, h.

(f) Consider the homogenous system $A\vec{x} = \vec{0}$. The nullity of A equals the number of free variables. Give an example of a matrix A with three pivot columns that has nullity 2.

(g) Let V be the vector space of all continuously differentiable vector functions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let S be the set of all vector solutions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the dynamical system

$$\begin{cases} x'(t) &= 2x(t) \\ y'(t) &= 4y(t) \end{cases}$$

Find two independent solutions $\vec{v}_1, \vec{v}_2$ such that $S = \textbf{span}(\vec{v}_1, \vec{v}_2)$. *This calculation proves that S is a subspace of V by Picard's theorem and the Span Theorem, hence S is a vector space.*

(h) The 4×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 & 3 \end{pmatrix}$$

Use this page to start your solution. Attach extra pages as needed.

Chapter 5

2. **(Chapter 5)** Do all parts.
(a) Solve for the general solution of $15y'' + 8y' + y = 0$.
(b) The characteristic equation is $r^2(2r+1)^3(r^2-2r+10) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.
(c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and xe^{3x}. Write a formula for the general solution.
(d) Mark with **X** the functions which **cannot** be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.

$e^{\ln\|2x\|}$	e^{x^2}	$2\pi + x$	$\cos(\ln\|x\|)$
$\cos(x\ln\|3.7125\|)$	$x^{-1}e^{-x}\sin(\pi x)$	$\cosh(x)$	$\sin^2(x)$

(e) Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 3x^2 + 10xe^{-x} + 4\cos(2x)$ is a solution.
(f) Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$(r^4 - 4r^3)((r - \ln(2))^2 + 4)^2 = 0.$$

(g) Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies.
(h) Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} = 11x^2 + 2x + 3 + 12\cos 2x + 13xe^{2x}$$

(i) Find a particular solution $y_p(x)$and the homogeneous solution $y_h(x)$ for $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 12x^2$.
(j) The differential equation $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 12x^2 + 6x$ has a particular solution $y_p(x)$ of the form $y = d_1x^2 + d_2x^3 + d_3x^4$. Find $y_p(x)$ by the method of undetermined coefficients (yes, find d_1, d_2, d_3).
(k) Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 2x = 5\sin(t)$.
(l) Find by variation of parameters an integral formula for a particular solution x_p of the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2+1)$. To save time, don't try to evaluate integrals (it's impossible).
(m) Write the solution $x(t)$ of

$$x''(t) + 25x(t) = 180\sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
To save time, don't convert to phase-amplitude form.

(n) Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

(o) Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

(p) Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

(q) Consider the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right).$$

Given the second order equation

$$2y''(x) + 4y'(x) + 3y(x) = 17\sin(x^2),$$

write the equations for the variables y_1, y_2, f.
To save time, do not compute W and do not write out y_p. Do not try to evaluate any integrals!

(r) A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots 0, 0, -1, -1, $2i$, $-2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 5e^{-x} + 4x^2 + x\cos 2x + \sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing subtracts credit.

(s) Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Use this page to start your solution. Attach extra pages as needed.

Chapter 10

3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(a) Display the details of Laplace's method to solve the system for $x(t)$. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + 3y, \\ y' &= -2y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

(b) Find $f(t)$ by partial fractions, the shifting theorem and the backward table, given

$$\mathcal{L}(f(t)) = \frac{2s^3 + 3s^2 - 6s + 3}{s^3(s-1)^2}.$$

(c) Solve for $f(t)$, given

$$\mathcal{L}(e^{2t}f(t)) + 2\frac{d^2}{ds^2}\mathcal{L}(tf(t)) = \frac{s+3}{(s+1)^3}.$$

(d) Solve for $f(t)$, given

$$\mathcal{L}(e^{-3t}f(t)) = \frac{s+1}{(s+2)^2}$$

(e) Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\frac{6}{s^4}$
$e^{-t}\cos(4t)$	
$(t+2)^2$	
t^2e^{-2t}	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\frac{3}{s^2+9}$	$\sin 3t$
$\frac{s-1}{s^2-2s+5}$	
$\frac{2}{(2s-1)^2}$	
$\frac{s}{(s-1)^3}$	

(f) Find $\mathcal{L}(f(t))$ from the Second Shifting theorem, given $f(t) = \sin(2t)\mathbf{u}(t-2)$, where $\mathbf{u}$ is the unit step function defined by $\mathbf{u}(t) = 1$ for $t \geq 0$, $\mathbf{u}(t) = 0$ for $t < 0$.

(g) Find $f(t)$ from the Second Shifting Theorem, given $\mathcal{L}(f(t)) = \dfrac{s\,e^{-\pi s}}{s^2+2s+17}$.

(h) Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \mathcal{L}(t\sin t)|_{s\to(s+2)}\,.$$

(i) Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s}$$

(j) Find $\mathcal{L}(f(t))$, given $f(t) = e^{2t}\left(\frac{\sin(t)}{t}\right)$.

(k) Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(l) Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\frac{\sin(t)}{t-\pi}$, where u is the unit step function.

(m) Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

(n) Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{(4)} + x^{(2)} = 3t + 4e^t + 5\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Use this page to start your solution. Attach extra pages as needed.

Chapter 6

4. (Chapter 6) Complete all parts.

(a) Define $E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find $E^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ without using matrix multiply.

(b) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 4 & 1 & 12 \\ -4 & 1 & -3 & 15 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & -2 & 7 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(c) Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, which has eigenvalues $1, 2, 2$, find all eigenvectors for eigenvalue 2.

(d) Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, 1, -1$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle those vectors from the list below which are possible columns of P.

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(e) Find the remaining eigenpairs of

$$E = \begin{pmatrix} 6 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

provided we already know one eigenpair

$$\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

(f) Suppose a 3×3 matrix A has eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(g) Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Display Fourier's model for the 3×3 matrix A.

(h) Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(i) Assume A is 2×2 and Fourier's model holds:

$$A\left(c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 2c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A.

(j) Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Circle the possible eigenvectors of A in the list below.

$$\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(k) Consider the 3×3 matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Show that matrix E has a Fourier model:

$$E\left(c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) = 4c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2c_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

(l) Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

(m) Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the matrix C.

(n) Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Then display Fourier's model for A.

Use this page to start your solution. Attach extra pages as needed.