KEY (Draft 1)

Differential Equations and Linear Algebra 2250

Midterm Exam 2

Exam Date: 17 April 2015 at 7:25am

Instructions: This in-class exam is designed to be completed in 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 4

Problem 1 (Linear Algebra) Do all parts.

1(a) [10%] For any matrix A, **nullity**(A) equals the number of free variables for the problem $A\vec{x} = \vec{0}$. How many $\underbrace{\text{pivot columns}}_{=\text{rank}(A)}$ in a 1000×54 matrix A with **nullity**$(A) = 21$? **Answer:** $\boxed{33}$

rank + Nullity = 54

1(b) [40%] Consider the following matrix and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -2 & -2 & 0 & 1 \\ 3 & -6 & -5 & 1 & 4 \\ -2 & 4 & 3 & -1 & -3 \end{pmatrix}, \quad \mathbf{rref}(A) = \begin{pmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b_1) Show all linear algebra steps used to solve $A\vec{x} = \vec{0}$ for $\vec{x}$.

(b_2) Report a basis for the solution space S of $A\vec{x} = \vec{0}$.

(b_3) Report the dimension of S.

(b_1) $\begin{pmatrix} 1 & -2 & -2 & 0 & 1 \\ 3 & -6 & -5 & 1 & 4 \\ -2 & 4 & 3 & -1 & -3 \end{pmatrix} \xrightarrow{\text{Combo}(1,2,-3)} \begin{pmatrix} 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ -2 & 4 & 3 & -1 & -3 \end{pmatrix} \xrightarrow{\text{Combo}(1,3,2)} \begin{pmatrix} 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix} \xrightarrow{\text{Combo}(2,3,1)} \begin{pmatrix} 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{\text{Combo}(3,1,2)} \begin{pmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ = last frame, or rref.

$\begin{cases} x_1 - 2x_2 + 2x_4 + 3x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \\ 0 = 0 \end{cases}$

Free vars $= x_2, x_4, x_5$ Lead $= x_1, x_3$

$\boxed{\text{vector sol} = \vec{x} = \begin{pmatrix} 2t_1 - 2t_2 - 3t_3 \\ t_1 \\ -t_2 - t_3 \\ t_2 \\ t_3 \end{pmatrix}}$

$\boxed{\begin{cases} x_1 = 2t_1 - 2t_2 - 3t_3 \\ x_2 = t_1 \\ x_3 = -t_2 - t_3 \\ x_4 = t_2 \\ x_5 = t_3 \end{cases}}$ Scalar Sol.

(b_2) partials on t_1, t_2, t_3 = Strang's special sols $= \boxed{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}$

S = Span(special sols)

(b_3) $\boxed{\text{DIM}(S) = 3}$, # of basis vectors.

Use this page to start your solution.

Name. ______________________________

Chapter 4

Problem 1 (Linear Algebra) Continued.

1(c) [20%] Check the independence tests which apply to prove that vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}$$

are independent in the vector space $\mathcal{R}^4$. Demerits are given for missing a box and also for checking the box for an inapplicable test.

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☒	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☒	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define $$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$ Then $\det(A) \neq 0$ implies independence of f, g, h.

Use this page to start your solution.

Name. ______________________________

Chapter 4

Problem 1 (Linear Algebra) Continued.

1(d) [30%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} -2 & 0 & -6 & 0 \\ 2 & 0 & 5 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}.$$

Extract from the list $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ a largest set of independent vectors.

$$\begin{pmatrix} -2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -6 & 5 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -2 & 2 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

combo(1,3,-3)
swap rows

rank = 2 = # pivots
pivots = 1, 2

$\boxed{\vec{v}_1, \vec{v}_2 \text{ is a largest independent set}}$

Use this page to start your solution.

Name. ________________________

Chapter 5

Problem 2 (Linear Differential Equations of Order n) Do all parts.

2(a) [20%] Mark with **X** the functions which **cannot** be a solution of a linear homogeneous differential equation with constant coefficients. Before marking, apply all algebraic simplifications. Test your choices against this theorem:

The general solution of a linear homogeneous nth order differential equation with constant coefficients is a linear combination of Euler solution atoms.

$1+x^2 \rightarrow$

$e^{\ln(1+x^2)}$	$e^{-x/2}$	$\ln(2)+x^2$	$\cos(\lvert x\rvert)$
$\sin(\pi x)$	$\frac{x}{1+x}$ X	$e^x + x^3$	$\tan(x)$ X

$\leftarrow \cos(x)$ or $\cos(-x)$, both $=\cos(x)$

2(b) [20%] Determine a **basis of solutions** for a homogeneous constant-coefficient linear differential equation, given the characteristic equation has roots $0, 0, 0; -1, -1; 1+2i, 1-2i, 1+2i, 1-2i$.

Basis $= 1, x, x^2;\ e^{-x}, xe^{-x};\ e^x\cos 2x, e^x\sin 2x, xe^x\cos 2x, xe^x\sin 2x$

Use this page to start your solution.

Name. ______________________

Chapter 5

Problem 2 (Linear Differential Equations of Order n) Continued.

2(c) [40%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 5x = 10\sin(t)$.

Und. Coeff. trial sol $= x = d_1 \cos t + d_2 \sin t$ by Rule I

Rule II does not apply. Substitute trial solution into DE above.

$$(-d_1 \cos t - d_2 \sin t) + 2(-d_1 \sin t + d_2 \cos t) + 5(d_1 \cos t + d_2 \sin t) = 10 \sin t$$

$$(-d_1 + 2d_2 + 5d_1)\cos t + (-d_2 - 2d_1 + 5d_2)\sin t = 10 \sin t$$

$$\begin{cases} 4d_1 + 2d_2 = 0 \\ -2d_1 + 4d_2 = 10 \end{cases}$$

$$d_1 = \frac{\begin{vmatrix} 0 & 2 \\ 10 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ -2 & 4 \end{vmatrix}} = \frac{-20}{20} = -1$$

$$d_2 = \frac{\begin{vmatrix} 4 & 0 \\ -2 & 10 \end{vmatrix}}{20} = \frac{40}{20} = 2$$

$x_p = d_1 \cos t + d_2 \sin t$

$x_p = -\cos t + 2 \sin t$

Because $\lim_{t=\infty} x_h = 0$, Then $\boxed{x_p = 2\sin t - \cos t}$ is

The steady-state periodic solution.

Must supply the reason for the steady-state answer.

Use this page to start your solution.

Name. ______________________

Chapter 5

Problem 2 (Linear Differential Equations of Order n) Continued.

2(d) [30%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots 0, 0, 0, 1, $1+2i$, $1-2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 3e^{-x}+4x^2+5xe^x\cos 2x+10e^x\sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.

To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$!

atoms for f: e^{-x}; $1, x, x^2$; $e^x\cos 2x, xe^x\cos 2x$; $e^x\sin 2x, xe^x\sin 2x$

NO Conflict — conflict — conflict — conflict

Rule II: atoms $= e^{-x}$; x^3, x^4, x^5; $xe^x\cos 2x, x^2e^x\cos 2x$; $xe^x\sin 2x, x^2e^x\sin 2x$

y = shortest trial sol = linear combination of the above Euler atoms

Use this page to start your solution.

Name. ______________________

Chapter 10

Problem 3. (Laplace Theory) Complete all parts.
It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

3(a) [20%] Display the s-fractions for $\mathcal{L}(x(t))$ and $\mathcal{L}(y(t))$ according to Laplace's Method.

$$\begin{cases} x' = 3x + 4y, \\ y' = -2y, \\ x(0) = 1, \quad y(0) = 2. \end{cases}$$

To save time, don't solve for $x(t)$ or $y(t)$!

$$\begin{cases} \mathcal{L}(x') = 3\,\mathcal{L}(x) + 4\mathcal{L}(y) \\ \mathcal{L}(y') = -2\,\mathcal{L}(y) \end{cases}$$

$$\begin{cases} s\mathcal{L}(x) - 1 = 3\,\mathcal{L}(x) + 4\,\mathcal{L}(y) \\ s\mathcal{L}(y) - 2 = -2\,\mathcal{L}(y) \end{cases}$$

$$\begin{cases} (s-3)\,\mathcal{L}(x) + (-4)\,\mathcal{L}(y) = 1 \\ (0)\,\mathcal{L}(x) + (s+2)\mathcal{L}(y) = 2 \end{cases}$$

$$\mathcal{L}(x) = \frac{\begin{vmatrix} 1 & -4 \\ 2 & s+2 \end{vmatrix}}{\begin{vmatrix} s-3 & -4 \\ 0 & s+2 \end{vmatrix}} = \frac{s+2+8}{(s-3)(s+2)} = \boxed{\frac{s+10}{(s-3)(s+2)}}$$

$$\mathcal{L}(y) = \frac{\begin{vmatrix} s-3 & 1 \\ 0 & 2 \end{vmatrix}}{(s-3)(s+2)} = \frac{2(s-3)}{(s-3)(s+2)} = \boxed{\frac{2}{s+2}}$$

Use this page to start your solution.

Name. ______________________________

Chapter 10

Problem 3. (Laplace Theory) Continued.

3(b) [30%] Solve for $f(t)$, given

$$\mathcal{L}(f(t))|_{s\to s-2} + 2\frac{d}{ds}\mathcal{L}(tf(t)) = \frac{s+1}{(s+2)^2}.$$

$$\mathcal{L}(e^{2t} f(t)) + \mathcal{L}(-2t^2 f(t)) = \frac{s+2}{(s+2)^2} + \frac{-1}{(s+2)^2}$$

$$\mathcal{L}((e^{2t} - 2t^2)f(t)) = \frac{1}{s+2} + \frac{-1}{(s+2)^2}$$

$$'' \quad = \mathcal{L}(e^{-2t}) + \left.\frac{-1}{s^2}\right|_{s\to s+2}$$

$$'' \quad = \mathcal{L}(e^{-2t}) + \mathcal{L}(-t)|_{s\to s+2}$$

$$= \mathcal{L}(e^{-2t}) + \mathcal{L}(-t e^{-2t})$$

$$= \mathcal{L}((1-t)e^{-2t})$$

Lerch's Thm $\Rightarrow$ $(e^{2t} - 2t^2)f(t) = (1-t)e^{-2t}$

$$\boxed{f(t) = \frac{(1-t)e^{-2t}}{e^{2t} - 2t^2}}$$

Use this page to start your solution.

Name. ______________________________

Chapter 10

Problem 3. (Laplace Theory) Continued.

3(c) [20%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\frac{6}{s^4}$
$\cos(4t)$	$\frac{s}{s^2+16}$
$(2t+1)^2$	$\frac{8}{s^3}+\frac{4}{s^2}+\frac{1}{s}$
t^2e^{2t}	$\frac{2}{(s-2)^3}$

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\frac{3}{s^2+9}$	$\sin 3t$
$\frac{s-2}{s^2-4s+5}$	$e^{2t}\cos(t)$
$\frac{1}{s^3}+\frac{1}{s+1}$	$\frac{t^2}{2}+e^{-t}$
$\frac{s+1}{(s-1)^2}$	$(1+2t)e^t$

$$(2t+1)^2 = 4t^2+4t+1 \to \frac{4\cdot 2}{s^3}+\frac{4}{s^2}+\frac{1}{s}$$

$$\mathcal{L}(t^2e^{2t}) = \mathcal{L}(t^2)\big|_{s\to s-2} = \frac{2}{s^3}\Big|_{s\to s-2} = \frac{2}{(s-2)^3}$$

$$\frac{s-2}{s^2-4s+5} = \frac{s-2}{(s-2)^2+1} = \frac{s}{s^2+1}\Big|_{s\to s-2} = \mathcal{L}(\cos t)\big|_{s\to s-2}$$
$$= \mathcal{L}(e^{2t}\cos t)$$

$$\frac{s+1}{(s-1)^2} = \frac{s-1}{(s-1)^2}+\frac{2}{(s-1)^2} = \frac{1}{s-1}+\frac{2}{(s-1)^2} = \left(\frac{1}{s}+\frac{2}{s^2}\right)\Big|_{s\to s-1}$$
$$= \mathcal{L}(1+2t)\big|_{s\to s-1} = \mathcal{L}((1+2t)e^t)$$

Use this page to start your solution.

Name. ______________________________

Chapter 10

Problem 3. (Laplace Theory) Continued.

3(d) [30%] Laplace's method is applied to the problem

$$\begin{cases} x'' + 2x' + x = e^t, \\ x(0) = 0, \ x'(0) = 1, \end{cases}$$

finding the answer $x(t) = \frac{1}{2}\sinh(t) + \frac{1}{2}te^{-t}$. Display the Laplace solution steps used to obtain the answer $x(t)$. Check the correctness of your answer with identity $\sinh(u) = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$.

$$\mathcal{L}(x'') + 2\mathcal{L}(x') + \mathcal{L}(x) = \mathcal{L}(e^t)$$

$$s\mathcal{L}(x') - x'(0) + 2(s\mathcal{L}(x) - x(0)) + \mathcal{L}(x) = \frac{1}{s-1}$$

$$s(s\mathcal{L}(x) - x(0)) - 1 + 2s\mathcal{L}(x) + \mathcal{L}(x) = \frac{1}{s-1}$$

$$(s^2 + 2s + 1)\mathcal{L}(x) = 1 + \frac{1}{s-1} = \frac{s-1+1}{s-1}$$

$$\mathcal{L}(x) = \frac{s-1+1}{(s-1)(s^2+2s+1)}$$

$$\mathcal{L}(x) = \frac{s}{(s-1)(s+1)^2}$$

$$\mathcal{L}(x) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$\mathcal{L}(x) = \mathcal{L}(Ae^t + Be^{-t} + Cte^{-t})$$

$$\boxed{x(t) = Ae^t + Be^{-t} + Cte^{-t}}$$

Clear fractions: $s = A(s+1)^2 + B(s-1)(s+1) + C(s-1)$

Sampling method

$s = 1$: $1 = 4A$

$s = -1$: $-1 = -2C$

$s = 0$: $0 = A + B(-1) + C(-1)$

$$\boxed{\begin{aligned} A &= \tfrac{1}{4} \\ C &= \tfrac{1}{2} \\ B &= -\tfrac{1}{4} \end{aligned}}$$

Use this page to start your solution.

Answer check: $\frac{1}{2}\sinh(t) + \frac{1}{2}te^{-t} = \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}te^{-t}$

$= Ae^t + Be^{-t} + Cte^{-t}$

$= x(t)$ ✓

Name. ______________________________

Chapter 6

Problem 4. (Eigenanalysis) Complete all parts.

4(a) [30%] Let $A = \begin{pmatrix} 6 & -4 & 4 \\ 8 & -6 & 11 \\ 0 & 0 & 5 \end{pmatrix}$. The eigenpairs are reported to be

$$\left(-2, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \left(2, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right), \left(5, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

Circle any incorrect eigenpairs.

$$\begin{pmatrix} 6 & -4 & 4 \\ 8 & -6 & 11 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6-8 \\ 8-12 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$ Eigenpair ✓

$$\begin{pmatrix} 6 & -4 & 4 \\ 8 & -6 & 11 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} \neq \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ Not an eigenpair

$$\begin{pmatrix} 6 & -4 & 4 \\ 8 & -6 & 11 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -6+11 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ +5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$ Eigenpair ✓

Use this page to start your solution.

Name. ______________________________

Chapter 6

Problem 4. (Eigenanalysis) Continued.

4(b) [40%] Find the eigenpairs of the matrix $A = \begin{pmatrix} -0.9 & 0.3 \\ 0.3 & -0.1 \end{pmatrix}$.

$$\begin{vmatrix} -.9-\lambda & .3 \\ .3 & -.1-\lambda \end{vmatrix} = 0 \rightarrow (.9+\lambda)(.1+\lambda) - (.3)^2 = 0$$

$$.09 + .1\lambda + .9\lambda + \lambda^2 - .09 = 0$$

$$\lambda^2 + \lambda = 0$$

$$\lambda(\lambda+1) = 0$$

$$\boxed{\lambda = 0, \lambda = -1}$$

$$\left(\begin{array}{cc|c} -9/10 & 3/10 & 0 \\ 3/10 & -1/10 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3/10 & -\frac{1}{10} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} 3x_1 - x_2 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = \frac{1}{3}t_1 \\ x_2 = t_1 \end{cases} \quad \boxed{\left(0, \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}\right) = \text{Eigenpair}}$$

$$\left(\begin{array}{cc|c} -\frac{9}{10}+1 & \frac{3}{10} & 0 \\ \frac{3}{10} & -\frac{1}{10}+1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} \frac{1}{10} & \frac{3}{10} & 0 \\ \frac{3}{10} & \frac{9}{10} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 3x_2 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = -3t_1 \\ x_2 = t_1 \end{cases} \quad \boxed{\left(-1, \begin{pmatrix} -3 \\ 1 \end{pmatrix}\right) = \text{Eigenpair}}$$

Use this page to start your solution.

Name. ______________________________

Chapter 6

Problem 4. (Eigenanalysis) Continued.

4(c) [30%] The vector general solution $\vec{u}(t)$ of a linear differential system $\frac{d}{dt}\vec{u} = A\vec{u}$ is given by a formula involving the eigenpairs of $n \times n$ diagonalizable matrix A.

Assume $A = \begin{pmatrix} -0.9 & 0.3 \\ 0.3 & -0.1 \end{pmatrix}$. Find the general solution $\vec{u}$ of the system $\frac{d}{dt}\vec{u} = A\vec{u}$.

$$\boxed{\vec{u} = c_1 e^{0t}\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

use eigenpairs from 4(b)

Use this page to start your solution.