Name KEY

Differential Equations and Linear Algebra 2250

SAMPLE Midterm Exam 1

Version 1, SOLUTIONS

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \frac{x^2-2}{1+x}$.

   (b) [25%] Solve $y' = \frac{1}{(\cos x+1)(\cos x-1)}$.

   (c) [25%] Solve $y' = \frac{(1+e^{2x})^2}{e^x}$, $y(0)=1$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\frac{d}{dt}\left(e^{2t}v(t)\right) = 20\bar{e}^{t}$, $v(0)=0$ and the position model $\frac{dx}{dt} = v(t)$, $x(0)=100$.

(a) $1+x=u$, $\frac{x^2-2}{1+x} = \frac{(u-1)^2-2}{u} = \frac{u^2-2u-1}{u} = u-2-1/u$

$y=\int y'dx = \int u\,dx = u^2/2 - 2u - \ln|u| + c = \boxed{\frac{(1+x)^2}{2} - 2x - \ln|1+x| + c_1}$

Also by long division, $\frac{x^2-2}{x+1} = x-1-1/(x+1) \Rightarrow \boxed{y = \frac{x^2}{2} - x - \ln|1+x| + c_2}$

(b) $(\cos x+1)(\cos x-1) = \cos^2 x - 1 = -\sin^2 x$

$y' = -\csc^2 x \Rightarrow y = \int y' = \int -\csc^2 x = \boxed{\cot x + c}$

(c) $y' = \frac{1}{e^x}(1+2e^{2x}+e^{4x}) = \bar{e}^{x} + 2e^x + e^{3x}$

$y = \int y' = \int(\bar{e}^{x} + 2e^x + e^{3x})\,dx = \boxed{-\bar{e}^{x} + 2e^x + \frac{1}{3}e^{3x} + c}$

(d) $e^{2t}v = \int \frac{d}{dt}(e^{2t}v)\,dt = \int 20\bar{e}^{t}dt = -20\bar{e}^{t} + c$

$v(0)=0 \Rightarrow 0 = -20+c \Rightarrow c = 20.$ $\boxed{v = -20e^{-3t} + 20e^{-2t}}$

$x = \int v = \int(-20e^{-3t} + 20e^{-2t})\,dt = \frac{20}{3}e^{3t} - 10e^{-2t} + c_1$

$x(0)=100 \Rightarrow 100 = \frac{20}{3} - 10 + c_1 \Rightarrow c_1 = \frac{310}{3}$

$\boxed{x(t) = \frac{20}{3}e^{-3t} - 10e^{-2t} + \frac{310}{3}}$

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2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided functions F and G exist such that $f(x,y) = F(x)G(y)$.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

☐ $y' + xy = y^2 + xy^2$	☒ $y' = (x-1)(y+1) + (y - xy)y$
☐ $y' = \cos(xy)$	☒ $e^x y' = x \ln\|y\| + x^2 \ln(y^2)$

(b) [10%] State a partial derivative test that concludes $y' = f(x,y)$ is a linear differential equation but not a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = xy^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + y\ln|x|$ is not separable. Supply all details.

(a) $y' = -xy + y^2(1+x)$ not sep

$y' = \cos(xy)$ not sep

$y' = xy - y + x - 1 + y^2 - xy^2$
$= (x-1)y + (1-x)y^2$
$= (x-1)(y-y^2)$ separable

$e^x y' = x\ln|y| + 2x^2 \ln|y|$
$= (x+2x^2)\ln|y|$ separable

(b) $\dfrac{\partial f}{\partial y} \neq 0$ and independent of y $\Rightarrow$ linear and not quadrature

(c) $f(x,y) = xy^2 \Rightarrow \dfrac{\partial f}{\partial y} = 2xy$ not independent of $y \Rightarrow$ Not Linear

(d) $f(x,y) = e^x + y\ln|x| \Rightarrow \dfrac{f_y}{f} = \dfrac{\ln|x|}{e^x + y\ln|x|}$ depends on x,
because $\left.\dfrac{f_y}{f}\right|_{x=1} = 0$ and $\left.\dfrac{f_y}{f}\right|_{x=2} = \dfrac{\ln 2}{e^2 + y\ln 2} \neq 0$

Therefore, $y' = f(x,y)$ is not separable.

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3. (Solve a Separable Equation)
Given $(yx + 2y)y' = ((2 + x)\sin(x)\cos(x) + x)(y^2 + 3y + 2)$.

(a) [80%] Find a non-constant solution in implicit form.
To save time, **do not solve** for y explicitly. No answer check expected.

(b) [20%] Find all constant solutions (also called equilibrium solutions; no answer check expected).

(a) $(x+2)yy' = [(x+2)\sin x \cos x + x](y+2)(y+1)$

$$\frac{yy'}{(y+2)(y+1)} = \sin x \cos x + \frac{x}{x+2}$$

$$\int LHS = \int \left(\frac{A}{y+2} + \frac{B}{y+1}\right)y' = A\ln|y+2| + B\ln|y+1| + c_1$$

$$\frac{y}{(y+2)(y+1)} = \frac{A}{y+2} + \frac{B}{y+1} \Rightarrow A = 2,\ B = -1$$

$$\int RHS = \int \sin x \cos x\, dx + \int \left(1 - \frac{2}{x+2}\right)dx$$

$$= \frac{\sin^2 x}{2} + x - 2\ln|x+2| + C_2$$

Implicit sol

$$\boxed{2\ln|y+2| - \ln|y+1| = \tfrac{1}{2}\sin^2 x + x - 2\ln|x+2| + c}$$

(b) Find y = constant in $y' = F(x)G(y)$ means $0 = F(x)G(y)$ or $G(y) = 0$. Here, $G(y) = \frac{(y+2)(y+1)}{y}$, so $\boxed{y=-1,\ y=-2}$

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Instructions: This in-class exam is 90 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)

(a) [50%] Solve the linear velocity model. Show all integrating factor steps.

$$\begin{cases} 5v'(t) &= -160 + \frac{12}{2t+1}v(t), \\ v(0) &= 80 \end{cases}$$

(b) [25%] Solve the homogeneous equation $\frac{dy}{dx} - \left(\frac{1}{x} + \cos(x)\right) y = 0.$

(c) [25%] Solve $4\frac{dy}{dx} - 24y = \frac{2}{\pi}$ using the superposition principle $y = y_h + y_p$. Expected are three answers, y_h, y_p and y.

(a) $5v' = -160 + \frac{12}{2t+1}v$

$v' = -32 + \frac{12}{10t+5}v$

(DE) $v' - \frac{12}{10t+5}v = -32$

$W = e^{\int p\,dt} = e^{-\frac{6}{5}\ln|10t+5|}$

$W = (10t+5)^{-6/5}$ (t near 0)

$\frac{(vW)'}{W} = -32$ Replace LHS of (DE)

$vW = -32\int W + c$ Quadr.

$= -32\frac{(10t+5)^{-1/5}}{(-1/5)(10)} + c$

$v = 16(10t+5) + c(10t+5)^{6/5}$

$v(0) = 80 \Rightarrow c = 0$

$v = 16(10t+5) = \boxed{160t + 80}$

ans check: IC ✓ DE ✓

$160 \overset{?}{=} -32 + 12(16)$ yes.

(b) $y = \frac{c}{W}$ Shortcut

$W = e^{\int p\,dx} = e^{-\ln|x| - \sin x}$

$y = \frac{cx}{e^{-\sin x}} = \boxed{cxe^{\sin x}}$

(c) Equil. Sol.: $y_p = \frac{-2}{24\pi}$

Homog. Sol.: $y' - 6y = 0$

$y_h = \frac{c}{e^{-6x}}$

Sol: $y = y_h + y_p$

$= \boxed{ce^{6x} - \frac{1}{12\pi}}$

ans check:

$4y' - 24y = 24ce^{6x} - 24y$

$= -24\left(\frac{-1}{12\pi}\right)$

$= \frac{2}{\pi}$ ✓

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5. (Stability)

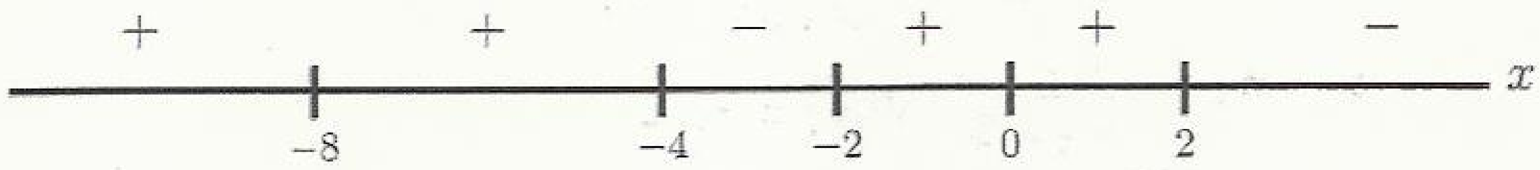
(a) [50%] Draw a phase line diagram for the differential equation

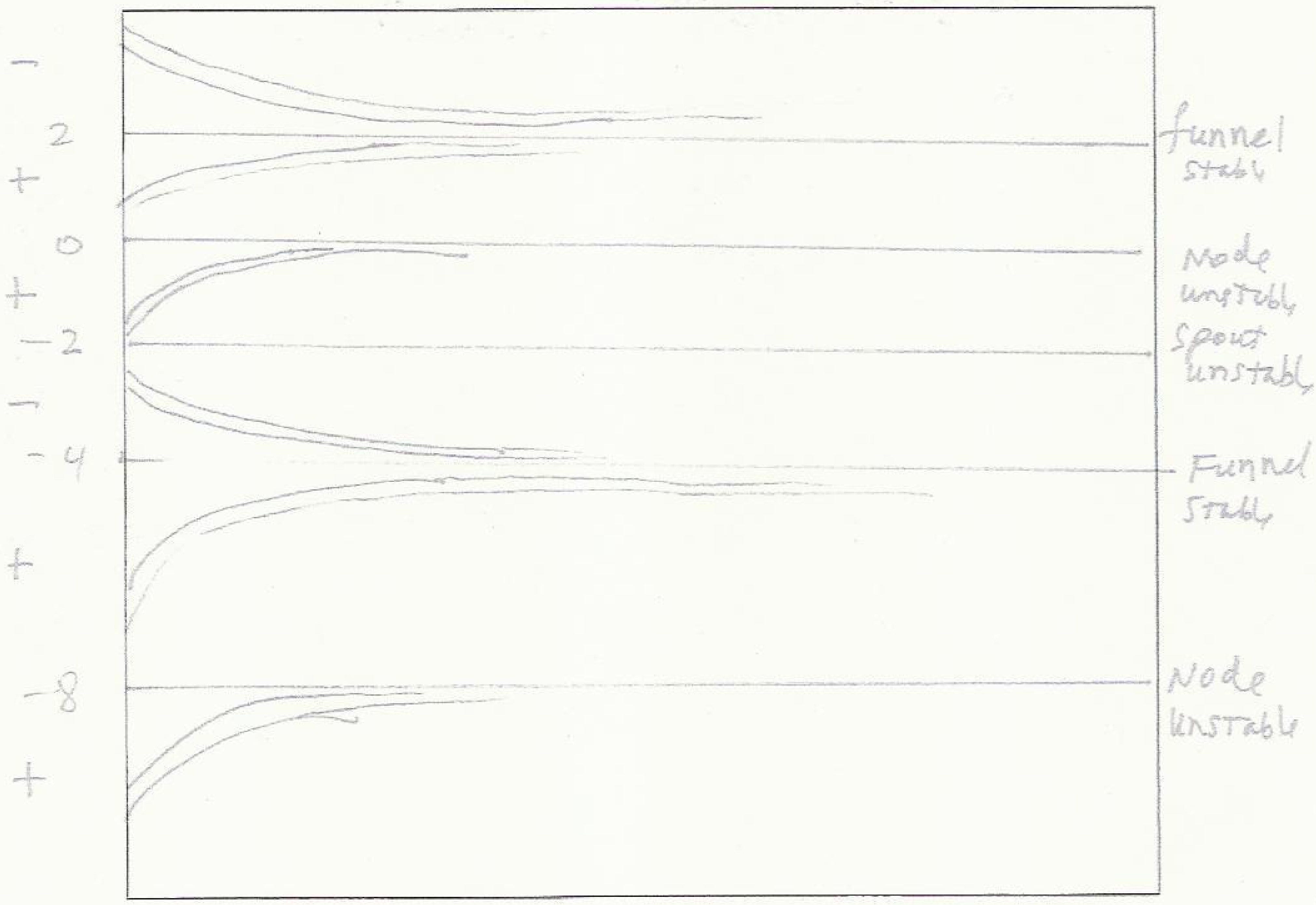
$$\frac{dx}{dt} = \sinh(x+1)\,(2-|4x-2|)^3\,(3-|x|)(x^2-9)(4-x^2).$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt. Definition: $\sinh(u) = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$.

roots $x = -1, 1, 0, 3, -3, 2, -2$

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.

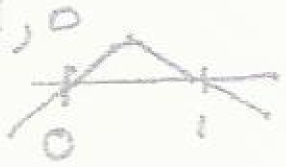




Calculations for problem 5 (a)

$\sinh(x+1) = 0 \Rightarrow x = -1$

$2 - |4x-2| = 0 \Rightarrow 4x-2 = \pm 2 \Rightarrow x = \frac{1}{2} \pm \frac{1}{2} = 1, 0$



$3 - |x| = 0 \Rightarrow x = \pm 3$

$x^2 - 9 = 0 \Rightarrow x = \pm 3$

$4 - x^2 = 0 \Rightarrow x = \pm 2$

$x = 4$: $+ \ - \ - \ + \ - \ = -$

$x = 2.5$ $= 5/2$: $+ \ - \ + \ - \ - \ = -$

$x = 3/2$ $= 1.5$: $+ \ - \ + \ - \ + \ = +$

$x = 1/2$: $+ \ + \ + \ - \ + \ = -$

$x = -1/2$: $+ \ - \ + \ - \ + \ = +$

$x = -3/2$: $- \ - \ + \ - \ + \ = -$

$x = -5/2$: $- \ - \ + \ - \ - \ = +$

$x = -4$: $- \ - \ - \ + \ - \ = +$

$\sinh(-\frac{1}{2}+1) = \sinh(\frac{1}{2}) > 0$

$\sinh(-\frac{1}{2}) = \frac{e^{-1/2} - e^{1/2}}{2} < 0$

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Instructions: This in-class exam is designed to be completed in 90 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

6. (The 3 Possibilities with Symbols)
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 0 & 0 & 0 \\ -2b-4 & 3 & a \\ b+1 & -1 & 0 \\ -1-b & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ b^2 \\ b \\ b^2-b \end{pmatrix}$$

(a) [40%] Determine a and b such that the system has a unique solution.

(b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

Because of a row of zeros (equation $0=0$), we can reduce the question to a 3×3 problem $A\vec{u} = \vec{B}$ where

$$A = \begin{pmatrix} -2b-4 & 3 & a \\ b+1 & -1 & 0 \\ -1-b & 1 & a \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} b^2 \\ b \\ b^2-b \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then $|A| = a - ab = a(1-b) \neq 0$ for $a \neq 0$ and $b \neq 1$

(a) unique sol when $|A| \neq 0$ (see reduction above) $\boxed{a \neq 0 \text{ and } b \neq 1}$

(b) when $a=0$, then $A = \begin{pmatrix} -2b-4 & 3 & 0 \\ b+1 & -1 & 0 \\ -1-b & 1 & 0 \end{pmatrix}$ has col rank ≤ 2
hence there is one free var, if the system is consistent. We do rref steps on $C = \langle A | B \rangle$ to get $\left(\begin{array}{ccc|c} 2b-4 & 3 & 0 & b^2 \\ b+1 & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \end{array}\right)$
There is a signal eq for $b \neq 0$,
hence no solution for $a=0, b\neq 0$.

(c) Define C as in part (b), then substitute $a=b=0$ to get rref step $\left(\begin{array}{ccc|c} -4 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. A homogeneous system is consistent. One free variable implies ∞-many solutions.

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Differential Equations and Linear Algebra 2250

Sample Midterm Exam 1

Instructions: This in-class exam is designed to be completed in under 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

7. **(Determinants)** Do all parts.

(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.

(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.

(c) [20%] Assume given 3×3 matrices A, B. Suppose $A^2B = E_2E_1A$ and E_1, E_2 are elementary matrices representing respectively a swap and a multiply by -5. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.

(d) [30%] Determine all values of x for which $(I+C)^{-1}$ fails to exist, where C equals the transpose of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 3x & 0 & 1 \\ x-1 & x & x \end{pmatrix}$.

(e) [30%] Let symbols a, b, c denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

(c) $|A^2B| = |E_2E_1A| \Rightarrow |A|^2|B| = |E_2||E_1||A|$ by The det product Thm. Then $|B|=10$, $|E_2|=-5$, $|E_1|=-1$ implies That $|A|^2(10) = (-5)(-1)|A|$ or $5(2|A|^2 - |A|) = 0$. So $|A|=0$ or $|A| = 1/2$. Then $|C| = |2A| = |2I||A| = 8|A| = \boxed{0 \text{ or } 4}$

(d) $|I+C^T| = |I^T + C^T| = |(I+C)^T| = |I+C| = \begin{vmatrix} 3 & 0 & -1 \\ 3x & 1 & 1 \\ x-1 & x & x+1 \end{vmatrix} =$
$(3)\begin{vmatrix} 1 & 1 \\ x & x+1 \end{vmatrix} + (-1)\begin{vmatrix} 3x & 1 \\ x-1 & x \end{vmatrix} = 3 - (3x^2 - x + 1) = x - 3x^2 + 2 =$
$(-3x-2)(x-1)$. Ans: $\boxed{x=1,\ x=-2/3}$

(e) entry $=$ cofactor$(A,4,3)/|A| = (-1)^{3+4}$minor$(A,4,3)/|A| = \boxed{1}$
minor$(A,4,3) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 1 \end{vmatrix} = -1$, $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 1 \end{vmatrix}(-1) = 1$ (cof on col 3)

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8. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

Vector Cross-product

(a) [10%] $\boxed{\text{True}}$ or false: There is a subspace S of $\mathcal{R}^3$ containing none of the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}3\\1\\2\end{pmatrix}$.

$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2 \Rightarrow \dim(S) = 1$, $S = \mathrm{Span}(\vec{v}_1 \times \vec{v}_2)$

(b) [10%] $\boxed{\text{True}}$ or false: The set of solutions $\vec{u}$ in $\mathcal{R}^3$ of a consistent matrix equation $A\vec{u} = \vec{b}$ can equal all vectors in the xy-plane, that is, all vectors of the form $\vec{u} = (x, y, 0)$.

Example: $\begin{cases} z = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$

(c) [10%] $\boxed{\text{True}}$ or false: Relations $x^2 + y^2 = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] $\boxed{\text{True}}$ or false: Equations $x = y$, $z = 2y$ define a subspace in $\mathcal{R}^3$. Kernel Theorem

(e) [20%] Linear algebra theorems are able to conclude that the set S of all polynomials $f(x) = c_0 + c_1 x + c_2 x^2$ such that $f'(x) + \int_0^1 f(x)x\,dx = 0$ is a vector space of functions. Explain why $V = \mathrm{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show S is a subspace of V. To save time, do not write any subspace proof details.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^5$ given by the system of restriction equations

$$\begin{aligned}
3x_1 + 2x_3 + 4x_4 + 10x_5 &= 0,\\
2x_1 + x_3 + 2x_4 + 4x_5 &= 0,\\
-2x_1 + 4x_5 &= 0,\\
2x_1 + 2x_3 + 4x_4 + 12x_5 &= 0.
\end{aligned}$$

(c) $x^2 + y^2 = 0 \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ which are 2 linear algebraic eqs.
Rest follows from The Kernel Theorem.

(e) $V = \mathrm{span}(1, x, x^2)$ = subspace of the vector space of all polynomials, by the SPAN Theorem (sec. 4.2)

S is a subspace by the Subspace Criterion (section 4.1)

Theorem S is a subspace of vector space V provided
(1) $\vec{0}$ is in S; (2) $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x} + \vec{y}$ in S; (3) $\vec{x}$ in S, c = const $\Rightarrow c\vec{x}$ in S

(f) Variable x_2 missing, so it's a free variable. Augmented Matrix is

$$C = \left(\begin{array}{ccccc|c} 3 & 0 & 2 & 4 & 10 & 0 \\ 2 & 0 & 1 & 2 & 4 & 0 \\ -2 & 0 & 0 & 0 & 4 & 0 \\ 2 & 0 & 2 & 4 & 12 & 0 \end{array}\right), \quad \mathrm{rref}(C) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Last frame alg scalar answer is

$$\begin{cases} x_1 = 2t_3 \\ x_2 = t_1 \\ x_3 = -2t_2 - 8t_3 \\ x_4 = t_2 \\ x_5 = t_3 \end{cases}$$

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$$\text{Basis} = \{\partial_{t_1}, \partial_{t_2}, \partial_{t_3}\} = \left\{ \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-8\\0\\1\end{pmatrix} \right\}$$