1. (Quadrature Equations)
   (a) [25%] Solve \( y' = \frac{x^2 - 2}{1 + x} \).
   (b) [25%] Solve \( y' = \frac{1}{(\cos x + 1)(\cos x - 1)} \).
   (c) [25%] Solve \( y' = \frac{1 + e^{2x}}{e^x}, \ y(0) = 1 \).
   (d) [25%] Find the position \( x(t) \) from the velocity model \( \frac{dx}{dt} = 90e^t, \ v(0) = 0 \) and the position model \( \frac{dx}{dt} = v(t), \ x(0) = 100 \).

\[ 1 + x = u, \ \frac{x^2 - 2}{1 + x} = \frac{u - 1}{u} = \frac{u^2 - 2u - 1}{u} = u - 2 - \frac{1}{u} \]
\[ y = \int y'dx = \int u\,dx = u\sqrt{2} - 2u - 8\ln|u| + C = \frac{(1 + x)^2 - 2x - 8\ln|1 + x| + C_1}{2} \]
Also by long division, \( \frac{x^2 - 2}{x + 1} = x - 1 - \frac{3}{x + 1} \) \[ y = \frac{x^2 - 2x - 8\ln|1 + x| + C_2}{2} \]

b) \( (\cos x + 1)(\cos x - i) = -\sin 2x \)
\[ y' = -\sin 2x \Rightarrow y = \int y'\,dx = -\cos 2x \Rightarrow y = \cos 2x + C \]

c) \( y' = \frac{1}{e^x}(1 + 2e^{2x} + e^{4x}) = e^{-x} + 2e^x + e^{3x} \)
\[ y = \int y'\,dx = \int (e^{-x} + 2e^x + e^{3x})\,dx = -e^{-x} + \frac{2e^x}{2} + \frac{e^{3x}}{3} + C \]

d) \( e^{2t} + 1 = \int \frac{d}{dt}(e^{2t} + 1)\,dt = \int 2e^{2t}\,dt = e^{2t} + C \)
\[ v(0) = 0 \Rightarrow 0 = -20 + C \Rightarrow C = 20 \]
\[ x = \int v\,dt = \int (-20e^{-2t} + 20e^{2t})\,dt = \frac{20}{3}e^{2t} + C_1 + 10e^{-2t} + C_1 \]
\[ x(0) = 100 \Rightarrow 100 = \frac{20}{3} + 10 + C_1 \Rightarrow C_1 = \frac{310}{3} \]
\[ x(t) = \frac{20}{3}e^{2t} - 10e^{-2t} + \frac{310}{3} \]

Use this page to start your solution. Attach extra pages as needed.
2. (Classification of Equations)

The differential equation \( y' = f(x,y) \) is defined to be \text{separable} provided functions \( F \) and \( C \) exist such that \( f(x,y) = F(x)C(y) \).

(a) [40%] Check \( \text{X} \) the problems that can be converted into separable form. No details expected.

- \( y' + xy = y^2 + xy^2 \)
- \( y' = \cos(xy) \)
- \( x^2 y' = x \ln |y| + x^2 \ln(y^2) \)

(b) [10%] State a partial derivative test that concludes \( y' = f(x,y) \) is a linear differential equation but not a quadrature differential equation.

(c) [20%] Apply classification tests to show that \( y' = xy^2 \) is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that \( y' = e^x + y \ln |x| \) is not separable. Supply all details.

\[ \begin{align*}
\frac{\partial f}{\partial y} \neq 0 \quad \text{and independent of } y & \Rightarrow \text{linear and not quadrature} \\
(\partial f / \partial y) y (x, y) & \Rightarrow \text{not independent of } y \Rightarrow \text{not linear} \\
\left. \frac{\partial y}{\partial x} \right|_{x=1} & = 0 \quad \text{and } \left. \frac{\partial y}{\partial x} \right|_{x=2} = \frac{\ln 2}{e^2 + y \ln 2} \neq 0 \\
\text{Therefore, } y' = f(x,y) & \text{ is not separable.}
\end{align*} \]
3. (Solve a Separable Equation)

Given \((gx + 2y)y' = (2 + x)\sin(x)\cos(x) + x(y^2 + 3y + 2)\).

(a) [80%] Find a non-constant solution in implicit form.

To save time, do not solve for \(y\) explicitly. No answer check expected.

(b) [20%] Find all constant solutions (also called equilibrium solutions; no answer check expected).

\[ (x+2)\frac{dy}{y} = \left[(x+2)\sin x \cos x + x \right] (y+2)(y+1) \]

\[ \frac{dy}{(y+2)(y+1)} = \sin x \cos x + \frac{x}{x+2} \]

\[ \int LHS = \int \left( \frac{A}{y+2} + \frac{B}{y+1} \right) dy = A \ln |y+2| + B \ln |y+1| + C \]

\[ \frac{y}{(y+2)(y+1)} = \frac{A}{y+2} + \frac{B}{y+1} \Rightarrow A = 2, B = -1 \]

\[ \int RHS = \int \sin x \cos x \, dx + \int (1 - \frac{2}{x+2}) \, dx \]

\[ = \sin x + C - 2 \ln |x+2| + C_2 \]

Implicit sol. \[ 2 \ln |y+2| - \ln |y+1| = \frac{1}{2} \sin^2 x + x - 2 \ln |x+2| + C \]

\[ \Box \]

\[ \text{Find } y = \text{constant in } y' = F(x) G(y) \text{ means } 0 = F(x) G(y) \]

or \( G(y) = 0 \). Here, \( G(y) = \frac{(y+2)(y+1)}{y} \), so \( y = -1, y = -2 \)

Use this page to start your solution. Attach extra pages as needed.
Differential Equations and Linear Algebra 2250
Sample Midterm Exam 1
Version 1

Instructions: This in-class exam is 90 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)
   (a) [50%] Solve the linear velocity model. Show all integrating factor steps.

\[
\begin{align*}
5v'(t) &= -160 + \frac{12}{2t+1} v(t), \\
v(0) &= 80
\end{align*}
\]

(b) [25%] Solve the homogeneous equation \( \frac{dy}{dx} - \left( \frac{1}{x} + \cos(x) \right) y = 0. \)

(c) [25%] Solve \( \frac{dy}{dx} - 2y = \frac{2}{x} \) using the superposition principle \( y = y_h + y_p. \) Expected are three answers, \( y_h, y_p \) and \( y. \)

\[\begin{array}{l}
\text{(a)} \quad 5v' = -160 + \frac{12}{2t+1} \quad v' = -32 + \frac{12}{10t+5} \\
\text{(DE)} \quad v' - \frac{12}{10t+5} v = -32 \\
W = e^{\int \frac{12}{10t+5} dt} = e^{\int \frac{12}{10t+5} dt} \\
W = (10t+5)^{-\frac{6}{5}} \\
W' = (10t+5)^{-\frac{6}{5}} \\
\text{LHS:} \quad \frac{dW}{dt} \\
W = -32 \quad \text{Quadr.} \\
W = -32 \left( 10t+5 \right)^{-\frac{6}{5}+c} \\
V = 16 \left( 10t+5 \right) + C \left( 10t+5 \right)^{6/5} \\
V(0) = 80 \Rightarrow C = 0 \\
V = 16 \left( 10t+5 \right) = \underline{160t+80} \\
\text{Ans check: IC } \checkmark \quad \text{DE } \checkmark \\
160 = -32 + 12 \left( 16 \right) \checkmark
\end{array}\]
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation
\[
\frac{dx}{dt} = \sinh(x + 1) (2 - |4x - 2|)^3 (3 - |x|)(x^2 - 9)(4 - x^2).
\]
Expected in the phase line diagram are equilibrium points and signs of \(dx/dt\). Definition:
\[
\sinh(x) = \frac{1}{2} e^x - \frac{1}{2} e^{-x}.
\]
Roots: \(x = -1, 1, 3, -3, 2, -2\)

(b) [50%] Assume an autonomous equation \(x'(t) = f(x(t))\). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

![Phase line diagram](image)
Calculations for problem 5(c)

\[ \sinh(x + 1) = 0 \Rightarrow x = -1 \]
\[ 2 - |4x - 2| = 0 \Rightarrow 4x - 2 = \pm 2 \Rightarrow x = \frac{1}{2} \pm \frac{1}{2} = 1, 0 \]
\[ 3 - 1x = 0 \Rightarrow x = \pm 3 \]
\[ x^2 - 9 = 0 \Rightarrow x = \pm 3 \]
\[ 4 - x^2 = 0 \Rightarrow x = \pm 2 \]

\[
x = 4: \quad + - - + - = \Large{\Theta}
\]
\[
x = 2.5: \quad + - + - = \Large{\Theta}
\]
\[
x = 3\frac{1}{2} \quad + - + - = \Large{\Theta}
\]
\[
x = 3: \quad + + + + = \Large{\Theta}
\]
\[
x < \frac{3}{2} \quad + - + - = \Large{\Theta}
\]
\[
x = \frac{3}{2}: \quad + - + - = \Large{\Theta}
\]
\[
x = \frac{3}{2}: \quad - + + - = \Large{\Theta}
\]
\[
x = -\frac{3}{2}: \quad - - + - = \Large{\Theta}
\]
\[
x = -3: \quad - - - - = \Large{\Theta}
\]

\[ \sinh(-\frac{3}{2}) = \sinh(\frac{3}{2}) > 0 \]
\[ \sinh(-\frac{3}{2}) = \frac{e^{-\frac{3}{2}} - e^{\frac{3}{2}}}{2} < 0 \]
Instructions: This in-class exam is designed to be completed in 90 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

6. (The 3 Possibilities with Symbols)
Let a, b and c denote constants and consider the system of equations

\[
\begin{pmatrix}
0 & 0 & 0 \\
-2b-4 & 3 & a \\
-b+1 & -1 & 0 \\
-1-b & 1 & a \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
b \\
a \\
-b^2-b \\
\end{pmatrix}
\]

(a) [40%] Determine a and b such that the system has a unique solution.

(b) [30%] Explain why \(a = 0\) and \(b \neq 0\) implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why \(a = b = 0\) implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

Because of a row of zeros (equation 0 = 0), we can reduce the question to a 3x3 problem \(\mathbf{A}\mathbf{u} = \mathbf{b}\) where

\[
\mathbf{A} = \begin{pmatrix}
-2b-4 & 3 & a \\
b+1 & -1 & 0 \\
-1-b & 1 & a \\
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}
b \\
b^2 \\
b^2+b \\
\end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
\]

\(\det \mathbf{A} = a - ab = a(1-b)\neq 0\) for \(a \neq 0\) and \(b \neq 1\)

(a) Unique sol when \(\det \mathbf{A} \neq 0\) (see reduction above)

\[
\begin{cases}
a \neq 0, a-b \\
b \neq 1
\end{cases}
\]

(b) When \(a = 0\), then \(\mathbf{A} = \begin{pmatrix}
-2b-4 & 3 & 0 \\
b+1 & -1 & 0 \\
-1-b & 1 & 0 \\
\end{pmatrix}\) has col rank \(\leq 2\)

hence \(\mathbf{A}\mathbf{u} = \mathbf{b}\) has no solution for \(b \neq 0\), hence no solution for \(a = 0, b \neq 0\).

(c) Define \(\mathbf{C} = \begin{pmatrix} \mathbf{A} \mid \mathbf{b} \end{pmatrix}\) to get

\[
\begin{pmatrix}
2b-4 & 3 & 0 & b^2 \\
b+1 & -1 & 0 & b^2 \\
0 & 0 & 0 & b^2 \\
\end{pmatrix}
\]

hence no solution for \(a = 0, b \neq 0\).

(c) Define \(\mathbf{C} = \begin{pmatrix} \mathbf{A} \mid \mathbf{b} \end{pmatrix}\) to get

\[
\begin{pmatrix}
-1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

A homogeneous system is consistent. One free variable implies \(\infty\) many solutions.

Use this page to start your solution. Attach extra pages as needed.
Name ___KEY___

Differential Equations and Linear Algebra 2250
Simple Midterm Exam 1

Instructions: This in-class exam is designed to be completed in under 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

7. (Determinants) Do all parts.
(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.
(b) [10%] True or False? The determinant of the negative of the n x n identity matrix is -1.
(c) [20%] Assume given 3 x 3 matrices A, B. Suppose $A^2B = E_3B_1A$ and $E_3, B_1, B_2$ are elementary matrices representing respectively a swap and a multiply by -5. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.
(d) [30%] Determine all values of $x$ for which $(I + C)^{-1}$ fails to exist, where $C$ equals the transpose of
   
   \[
   \begin{pmatrix}
   2 & 0 & -1 \\
   3x & 0 & 1 \\
   x-1 & x & x
   \end{pmatrix}
   \]
   
   (e) [30%] Let symbols $a, b, c$ denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of $A^{-1}$, given $A$ below.
   
   \[
   A = \begin{pmatrix}
   1 & 1 & 0 & 0 \\
   1 & 0 & 0 & 0 \\
   a & b & 0 & 1 \\
   1 & c & 1 & 2
   \end{pmatrix}
   \]

\[\begin{array}{l}
\text{(c) } |A^2B| = |E_3E_1A| \Rightarrow |A|^2|B| = |E_3| |E_1| |A| \text{ by The det product rule. Then } |B| = 10, |E_3| = -5, |E_1| = -1 \text{ imply that } |A|^2(10) = (-5)(-1)|A| \text{ or } 5(2|A|^2 - |A|) = 0. \text{ So } |A| = 0 \text{ or } |A| = 1/2. \text{ Then } |C| = 12|A| = 12|A| |A| = 8|A| = 0 \text{ or } 4.
\end{array}\]

\[\begin{array}{l}
\text{(d) } |I + C^T| = |I + C^T| = |I + C^T| = |I + C| = |3 0 -1 1| = \\
\text{Ans: } x = 1, x = -2/3
\end{array}\]

\[\begin{array}{l}
\text{(c) } \text{entry } = \text{cofactor} (A_{ij}, 3)/|A| = (-1)^{i+j} \text{minor} (A_{ij}, 3)/|A| = 1 \\
\text{minor} (A_{ij}, 3) = \begin{vmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & b & 1
\end{vmatrix} = -1 \Rightarrow |A| = \begin{vmatrix}
1 & 0 & 0 \\
0 & b & 1 \\
0 & 0 & c
\end{vmatrix} = 1 \text{ (cofactor cof on cof)}.
\end{array}\]

Use this page to start your solution. Attach extra pages as needed.
P. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace $S$ of $\mathbb{R}^3$ containing none of the vectors $V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $V_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $V_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(b) [10%] True or false: The set of solutions $S$ in $\mathbb{R}^3$ of a consistent matrix equation $A\mathbf{u} = \mathbf{b}$ can equal all vectors in the $xy$-plane, $\mathbf{u} = (x, y, 0)$, if $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

(c) [10%] True or false: Relations $x^2 + y^2 = 0, y + z = 0$ define a subspace in $\mathbb{R}^3$.

(d) [10%] True or false: Equations $x = y, z = x$ define a subspace in $\mathbb{R}^3$.

(e) [20%] Linear algebra theorems are able to conclude that the set $S$ of all polynomials $p(x) = t_0 + c_1 x + c_2 x^2$ such that $p(x)\int_0^x p(x)dx = 0$ is a vector space of functions. Explain why $V = \text{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show $S$ is a subspace of $V$. To save time, do not write any subspace proof details.

(f) [40%] Find a basis for the subspace of $\mathbb{R}^3$ given by the system of restriction equations

\[
\begin{align*}
3x_1 + 2x_2 + 4x_3 + 10x_4 &= 0, \\
2x_1 + x_2 + 2x_3 + 4x_4 &= 0, \\
-2x_1 + 2x_2 + 4x_3 &= 0, \\
2x_2 + 4x_3 + 12x_4 &= 0.
\end{align*}
\]

\[x^2 + y^2 = 0 \iff \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{which are two linear algebraic Eq.}
\]

Rest follows from the Kernel Theorem.

\[V = \text{span}(1, x, x^2) = \text{subspace of the vector space of all polynomials, by the Span Theorem (Sec. 4.2)}.
\]

\[S \text{ is a subspace by the Subspace Criterion (Section 4.1).}
\]

\[\text{Theorem: } S \text{ is a subspace of vector space } V \text{ provided}
\]

\[(1) \quad \text{0 is in } S \qquad \text{(2) } x, y \text{ in } S \Rightarrow x + y \text{ in } S \qquad \text{(3) } x \text{ in } S, c \text{ in } \mathbb{C} \Rightarrow cx \text{ in } S
\]

\[\text{Variable } x_2 \text{ missing, so it's a free variable. Augmented matrix}
\]

**C = \begin{pmatrix} 3 & 0 & 2 & 4 & 10 \\ 2 & 0 & 1 & 2 & 4 \\ -2 & 0 & 2 & 4 & 12 \end{pmatrix}**

\[\text{Rank(C)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[\text{Last Frame alg. Step}
\]

\[X_1 = 2t_3
\]

\[X_2 = t_3
\]

\[X_4 = -2t_1 - 2t_3
\]

\[X_5 = t_3
\]

**Use this page to start your solution. Attach extra pages as needed.**

\[\text{Basis} = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

\[\text{Last Frame alg. Step}
\]

\[X_1 = 2t_3
\]

\[X_2 = t_3
\]

\[X_4 = -2t_1 - 2t_3
\]

\[X_5 = t_3
\]