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Differential Equations and Linear Algebra 2250

Sample Midterm Exam 1

Version 1, 19 Feb 2015

Instructions: This in-class exam is 90 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [25%] Solve $y' = \dfrac{x^2-2}{1+x}$.

(b) [25%] Solve $y' = \dfrac{1}{(\cos x+1)(\cos x-1)}$.

(c) [25%] Solve $y' = \dfrac{(1+e^{2x})^2}{e^x}$, $y(0)=1$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{2t}v(t)\right) = 20e^{-t}$, $v(0)=0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0)=100$.

Answers: (a) $y(x) = (1/2)x^2 - x - \ln(1+x) + C$, (b) $y = \cot(x) + C$, (c) $y(x) = -e^{-x} + 2\,e^x + \frac{1}{3}\,e^{3x} - \frac{1}{3}$ (d) $v(t) = -20\,e^{-3t} + 20\,e^{-2t}$, $x(t) = \frac{20}{3}\,e^{-3t} - 10\,e^{-2t} + \frac{310}{3}$.

2. (Classification of Equations)

The differential equation $y' = f(x,y)$ is defined to be **separable** provided functions F and G exist such that $f(x,y) = F(x)G(y)$.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be converted into separable form. No details expected.

☐	$y' + xy = y^2 + xy^2$	☐	$y' = (x-1)(y+1) + (y-xy)y$
☐	$y' = \cos(xy)$	☐	$e^x y' = x\ln\lvert y\rvert + x^2\ln(y^2)$

(b) [10%] State a partial derivative test that concludes $y' = f(x,y)$ is a linear differential equation but not a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = xy^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + y\ln\lvert x\rvert$ is not separable. Supply all details.

3. (Solve a Separable Equation)

Given $(yx+2y)y' = ((2+x)\sin(x)\cos(x) + x)\,(y^2+3y+2)$.

(a) [80%] Find a non-constant solution in implicit form.
To save time, **do not solve** for y explicitly. No answer check expected.

(b) [20%] Find all constant solutions (also called equilibrium solutions; no answer check expected).

4. (Linear Equations)

(a) [50%] Solve the linear model $4x'(t) = -160 + \dfrac{24}{2t+3}x(t)$, $x(0) = 30$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\dfrac{dy}{dx} - (\cos x)y = 0$.

(c) [30%] Solve $11\dfrac{dy}{dx} + 33y = 5$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

Answers reported after the exam: (a) $x(t) = 20t + 30$, (b) $y(x) = c\,e^{\sin(x)}$, (c) $y(x) = \frac{5}{33} + c\,e^{-3x}$

5. (Stability)

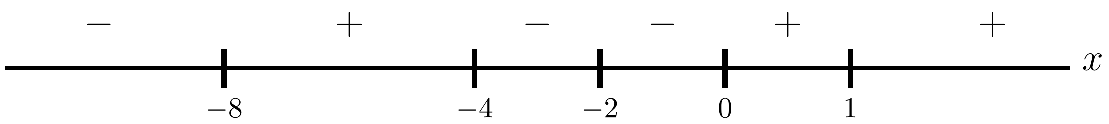
(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+x^4)\,(2 - |3x-1|)^3\,(2+x)(x^2-4)(1-x^2)^5.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

Answers: Roots $-2, -1, -1/3, 0, 1, 2$. Signs left to right are:
MINUS, MINUS, PLUS, MINUS, MINUS, MINUS, PLUS.

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.



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6. (The 3 Possibilities with Symbols)

Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 0 & 0 & 0 \\ -2b-4 & 3 & a \\ b+1 & -1 & 0 \\ -1-b & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ b^2 \\ b \\ b^2 - b \end{pmatrix}$$

(a) [40%] Determine a and b such that the system has a unique solution.

(b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

7. (Determinants) Do all parts.

(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.

(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.

(c) [20%] Assume given 3×3 matrices A, B. Suppose $A^2B = E_2E_1A$ and E_1, E_2 are elementary matrices representing respectively a swap and a multiply by -5. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.

(d) [30%] Determine all values of x for which $(I + C)^{-1}$ fails to exist, where C equals the transpose of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 3x & 0 & 1 \\ x-1 & x & x \end{pmatrix}$.

(e) [30%] Let symbols a, b, c denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

8. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace S of $\mathcal{R}^3$ containing none of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

(b) [10%] True or false: The set of solutions $\vec{u}$ in $\mathcal{R}^3$ of a consistent matrix equation $A\vec{u} = \vec{b}$ can equal all vectors in the xy-plane, that is, all vectors of the form $\vec{u} = (x, y, 0)$.

(c) [10%] True or false: Relations $x^2 + y^2 = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] True or false: Equations $x = y$, $z = 2y$ define a subspace in $\mathcal{R}^3$.

(e) [20%] Linear algebra theorems are able to conclude that the set S of all polynomials $f(x) = c_0 + c_1x + c_2x^2$ such that $f'(x) + \int_0^1 f(x)x\,dx = 0$ is a vector space of functions. Explain why $V = \mathbf{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show S is a subspace of V. To save time, do not write any subspace proof details.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^5$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 2x_3 & + & 4x_4 & + & 10x_5 & = & 0, \\ 2x_1 & + & x_3 & + & 2x_4 & + & 4x_5 & = & 0, \\ -2x_1 & & & & & + & 4x_5 & = & 0, \\ 2x_1 & + & 2x_3 & + & 4x_4 & + & 12x_5 & = & 0. \end{array}$$