Derivation of the Laplacian in Polar Coordinates

We suppose that u is a smooth function of x and y, and of r and θ. We will show that

$$u_{xx} + u_{yy} = u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} \tag{1}$$

and

$$|u_x|^2 + |u_y|^2 = |u_r|^2 + (1/r^2)|u_\theta|^2. \tag{2}$$

We assume that our functions are always nice enough to make mixed partials equal: $u_{xy} = u_{yx}$, etc.

The chain rule says that, for any smooth function ψ,

$$\psi_x = \psi_r r_x + \psi_\theta \theta_x$$
$$\psi_y = \psi_r r_y + \psi_\theta \theta_y.$$

Now,

$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1}(y/x) + c,$$

(where the constant c depends on the quadrant). Therefore, after differentiating and doing some algebra,

$$r_x = \cos\theta$$
$$r_y = \sin\theta$$
$$\theta_x = \frac{-\sin\theta}{r}$$
$$\theta_y = \frac{\cos\theta}{r},$$

implying

$$\psi_x = (\cos\theta)\psi_r + \frac{-\sin\theta}{r}\psi_\theta \tag{3}$$
$$\psi_y = (\sin\theta)\psi_r + \frac{\cos\theta}{r}\psi_\theta. \tag{4}$$

Formula (2) is now easy. (Apply the last two equations to $\psi = u$.)

If we apply equation (3) to $\psi = u_x$, we get:

$$u_{xx} = (\cos\theta)_x u_r + (\cos\theta)u_{xr} + (\frac{-\sin\theta}{r})_x u_\theta + \frac{-\sin\theta}{r}u_{x\theta}.$$

Applying (3) to $\psi = \cos\theta$ and $\psi = \frac{-\sin\theta}{r}$, we get

$$\begin{aligned}
(\cos\theta)_x &= (\cos\theta)\cdot 0 + \left(\frac{-\sin\theta}{r}\right)(-\sin\theta) \\
&= \frac{\sin^2\theta}{r}; \\
\left(\frac{-\sin\theta}{r}\right)_x &= \frac{\cos\theta\sin\theta}{r^2} + \left(\frac{-\sin\theta}{r}\right)\left(\frac{-\cos\theta}{r}\right) \\
&= \frac{2\cos\theta\sin\theta}{r^2}.
\end{aligned}$$

Applying it to $\psi = u_r$ and $\psi = u_\theta$, we get

$$\begin{aligned}
u_{rx} &= (\cos\theta)u_{rr} + \left(\frac{-\sin\theta}{r}\right)u_{r\theta} \\
u_{\theta x} &= (\cos\theta)u_{r\theta} + \left(\frac{-\sin\theta}{r}\right)u_{\theta\theta},
\end{aligned}$$

implying

$$u_{xx} = (\cos^2\theta)u_{rr} + \left(\frac{\sin^2\theta}{r}\right)u_r - 2\left(\frac{\cos\theta\sin\theta}{r}\right)u_{r\theta} + 2\left(\frac{\cos\theta\sin\theta}{r^2}\right)u_\theta + \left(\frac{\sin^2\theta}{r^2}\right)u_{\theta\theta}. \tag{5}$$

Similarly, if we apply (4) to $\psi = u_y$, we get:

$$u_{yy} = (\sin\theta)_y u_r + (\sin\theta)u_{yr} + \left(\frac{\cos\theta}{r}\right)_y u_\theta + \left(\frac{\cos\theta}{r}\right)u_{\theta y}.$$

Now we apply (4) to $\psi = \sin\theta$ and $\psi = \frac{\cos\theta}{r}$ and get:

$$\begin{aligned}
(\sin\theta)_y &= (\sin\theta)\cdot 0 + \left(\frac{\cos\theta}{r}\right)\cos\theta \\
&= \frac{\cos^2\theta}{r} \\
\left(\frac{\cos\theta}{r}\right)_y &= \left(\frac{-\sin\theta\cos\theta}{r^2}\right) - \left(\frac{\sin\theta\cos\theta}{r^2}\right) \\
&= \frac{-2\sin\theta\cos\theta}{r^2}.
\end{aligned}$$

If we apply (4) to $\psi = u_r$ and $\psi = u_\theta$ we get:

$$u_{ry} = (\sin\theta)u_{rr} + \left(\frac{\cos\theta}{r}\right)u_{r\theta}$$
$$u_{\theta y} = (\sin\theta)u_{r\theta} + \left(\frac{\cos\theta}{r}\right)u_{\theta\theta}.$$

Plugging these all in, we get

$$u_{yy} = (\sin^2\theta)u_{rr} + \left(\frac{\cos^2\theta}{r}\right)u_r + 2\left(\frac{\cos\theta\sin\theta}{r}\right)u_{r\theta} - 2\left(\frac{\cos\theta\sin\theta}{r^2}\right)u_\theta + \left(\frac{\cos^2\theta}{r^2}\right)u_{\theta\theta}. \tag{6}$$

If we add equations (5) and (6), exactly the RIGHT THINGS cancel, and exactly the RIGHT THINGS add up to 1, and we get (1). We put the two equations on top of each other to make this clearer:

$$u_{xx} = (\cos^2\theta)u_{rr} + \left(\frac{\sin^2\theta}{r}\right)u_r - 2\left(\frac{\cos\theta\sin\theta}{r}\right)u_{r\theta} + 2\left(\frac{\cos\theta\sin\theta}{r^2}\right)u_\theta + \left(\frac{\sin^2\theta}{r^2}\right)u_{\theta\theta}$$
$$u_{yy} = (\sin^2\theta)u_{rr} + \left(\frac{\cos^2\theta}{r}\right)u_r + 2\left(\frac{\cos\theta\sin\theta}{r}\right)u_{r\theta} - 2\left(\frac{\cos\theta\sin\theta}{r^2}\right)u_\theta + \left(\frac{\cos^2\theta}{r^2}\right)u_{\theta\theta}.$$