Periodic Functions and Orthogonal Systems

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Periodic Functions

Definition. A function f is T-periodic if and only if $f(t+T) = f(t)$ for all t.

Definition. The **floor** function is defined by

$$\mathsf{floor}(x) = \text{greatest integer not exceeding } x.$$

Theorem. Every function g defined on $0 \le x \le T$ has a T-periodic extension f defined on the whole real line by the formula

$$f(x) = g(x - T\,\mathsf{floor}(x/T)).$$

Even and Odd Functions

Definition. A function $f(x)$ is said to be *even* provided

$$f(-x) = f(x), \quad \text{for all } x.$$

A function $g(x)$ is said to be *odd* provided

$$g(-x) = -g(x), \quad \text{for all } x.$$

Definition. Let $h(x)$ be defined on $[0, T]$.
The **even extension** f of h to $[-T, T]$ is defined by

$$f(x) = \begin{cases} h(x) & 0 \le x \le T, \\ h(-x) & -T \le x < 0. \end{cases}$$

Assume $h(0) = 0$. The **odd extension** g of h to $[-T, T]$ is defined by

$$g(x) = \begin{cases} h(x) & 0 \le x \le T, \\ -h(-x) & -T \le x < 0. \end{cases}$$

Properties of Even and Odd Functions

Theorem. Even and odd functions have the following properties.

- The product and quotient of an even and an odd function is odd.
- The product and quotient of two even functions is even.
- The product and quotient of two odd functions is even.
- Linear combinations of odd functions are odd.
- Linear combinations of even functions are even.

Theorem. Among the trigonometric functions, the cosine and secant are even and the sine and cosecant, tangent and cotangent are odd.

Properties of Periodic Functions

Theorem. If f is T-periodic and continuous, and a is any real number, then

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$$

Theorem. If f and g are T-periodic, then

- $c_1 f(x) + c_2 g(x)$ is T-periodic for any constants c_1, c_2
- $f(x)g(x)$ is T-periodic
- $f(x)/g(x)$ is T-periodic
- $h(f(x))$ is T-periodic for any function h

Piecewise-Defined Functions

Definition. For $a \leq b$, define $\mathrm{pulse}(x, a, b) = \begin{cases} 1 & a \leq x < b, \\ 0 & \text{otherwise.} \end{cases}$

Definition. Assume that $a \leq x_1 \leq x_2 \leq \cdots \leq x_{n+1} \leq b$. Let f_1, f_2, …, f_n be continuous functions defined on $-\infty < x < \infty$. A **piecewise continuous function** f on a closed interval $[a, b]$ is a sum

$$f(x) = \sum_{j=1}^{n} f_j(x)\,\mathrm{pulse}(x, x_j, x_{j+1}).$$

If additionally f_1, …, f_n are continuously differentiable on $-\infty < x < \infty$, then sum f is called a **piecewise continuously differentiable** function.

Representations of Even and Odd Extensions

Theorem. The following formulas are valid.

- If f is the even extension on $[-T, T]$ of a function g defined on $[0, T]$, then

$$f(x) = g(x)\,\mathrm{pulse}(x, 0, T) + g(-x)\,\mathrm{pulse}(x, -T, 0).$$

- If f is the odd extension on $[-T, T]$ of a function h defined on $[0, T]$, then

$$f(x) = h(x)\,\mathrm{pulse}(x, 0, T) - h(-x)\,\mathrm{pulse}(x, -T, 0).$$

- The $2T$-periodic extension F of f is given by

$$F(x) = f(x - 2T\,\mathsf{floor}(x/(2T))).$$

Integration and Differentiation of Piecewise-Defined Functions

Theorem. Assume the piecewise-defined function is given on $[a, b]$ by the pulse formula

$$f(x) = \sum_{j=1}^{n} f_j(x)\,\mathrm{pulse}(x, x_j, x_{j+1}).$$

Then

$$\int_a^b f(x)dx = \sum_{j=1}^{n} \int_{x_j}^{x_{j+1}} f_j(x)dx.$$

If x is not a division point $x_1, \ldots, x_{n+1}$, and each f_j is differentiable, then

$$f'(x) = \sum_{j=1}^{n} f_j'(x)\,\mathrm{pulse}(x, x_j, x_{j+1}).$$

Inner Product

Definition. Define the inner product symbol $\langle f, g\rangle$ by the formula

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

If the interval $[a, b]$ is important, then we write $\langle f, g\rangle_{[a,b]}$.

The **inner product** $\langle \cdot, \cdot\rangle$ has the following properties:

- $\langle f, f\rangle \geq 0$ and for continuous f, $\langle f, f\rangle = 0$ implies $f = 0$.
- $\langle f, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$
- $c\langle f, g\rangle = \langle cf, g\rangle$
- $\langle f, g\rangle = \langle g, f\rangle$

Orthogonal Functions

Definition. Two nonzero functions f, g defined on $a \leq x \leq b$ are said to be **orthogonal** provided $\langle f, g \rangle = 0$.

Definition. Functions $f_1, \ldots, f_n$ are called an **orthogonal system** provided

- $\langle f_j, f_j \rangle > 0$ for $j = 1, \ldots, n$
- $\langle f_i, f_j \rangle = 0$ for $i \neq j$

Theorem. An orthogonal system $f_1, \ldots, f_n$ on $[a, b]$ is linearly independent on $[a, b]$.

Theorem. The first three Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(x^2 - 1)$ are an orthogonal system on $[-1, 1]$. In general, the system $\{P_j(x)\}_{j=0}^{\infty}$ is orthogonal on $[-1, 1]$.

Theorem. The trigonometric system 1, $\cos x$, $\cos 2x$, $\ldots$, $\sin x$, $\sin 2x$, $\ldots$ is an orthogonal system on $[-\pi, \pi]$.

Trigonometric System Details

Theorem. The orthogonal trigonometric system $\mathbf{1}$, $\mathbf{\cos x}$, $\mathbf{\cos 2x}$, …, $\mathbf{\sin x}$, $\mathbf{\sin 2x}$, … on $[-\pi, \pi]$ has the orthogonality relations

$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \\
\langle \cos nx, \cos mx \rangle &= \int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m > 0 \\ 2\pi & n = m = 0 \end{cases} \\
\langle \sin nx, \cos mx \rangle &= \int_{-\pi}^{\pi} \sin nx \cos mx dx = 0.
\end{aligned}$$