Fourier Transform for Partial Differential Equations

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Introduction: Fourier Transform

The Fourier transform creates another representation of a signal, specifically a representation as a weighted sum of complex exponentials. It is designed for non-periodic signals that decay at infinity, the condition that $\int_{-\infty}^{\infty} |f(x)| dx$ is finite.

Because of Euler's formula

$$e^{iq} = \cos(q) + i \sin(q)$$

where $i^2 = -1$, the Fourier transform produces a representation of a signal (or an image) as a weighted sum of sines and cosines.

Given a signal (or image) a and its Fourier transform A, then the **forward Fourier transform** goes from the spatial domain, either continuous or discrete, to the frequency domain, which is always continuous. The **inverse Fourier transform** goes from the frequency domain back to the spatial domain.

$$\text{Forward}: \quad A = F(a), \quad \text{Inverse}: \quad a = F^{-1}(A)$$

Definition: Fourier Transform

$$\boldsymbol{F(w) = FT[f](w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{iwx}dx}$$

$$\boldsymbol{f(x) = FT[f]^{-1}(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx}dw}$$

The **Reciprocity Relation** connects the two similar formulas:

$$\begin{aligned} \boldsymbol{f(-u)} &= \boldsymbol{\int_{-\infty}^{\infty} F(w)e^{iwu}dw} \\ &= \boldsymbol{2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{iux}dx} \\ &= \boldsymbol{2\pi FT[F](u)} \end{aligned}$$

In words, setting $\boldsymbol{x = -u}$ in the inverse Fourier transform equation produces the forward Fourier transform equation for the function $\boldsymbol{F}$ multiplied by $\boldsymbol{2\pi}$.

Fourier Transform Properties

1. Linearity $FT[f+g] = FT[f] + FT[g]$ and $FT[cf] = cFT[f]$
2. x-differentiation $FT[f'] = (-iw)FT[f]$
3. w-differentiation $FT[xf(x)] = -i\dfrac{d}{dw}FT[f]$
4. Convolution $FT[f*g] = FT[f]FT[g]$, where

   $f*g(x) = g*f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-t)g(t)dt$
5. x-shifting $FT[f(x-a)] = e^{iaw}FT[f]$
6. w-shifting $FT[e^{iax}f(x)](w) = FT[f](w+a)$

Parseval's Energy Identity

The square $|f(t)|^2$ of the time signal represents how the energy contained in the signal distributes over time t, while the spectrum squared $|F(w)|^2$ represents how the energy distributes over frequency (the power density spectrum). The same amount of energy is contained in either time or frequency domain, because of Parseval's formula:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(t)|^2dt = \int_{-\infty}^{\infty}|F(w)|^2dw$$

To evaluate what this means graphically, compute both integrands and graph them on a large interval. Use the Gaussian example

$$f(t) = e^{-\alpha t^2}$$

Fourier Sine and Cosine Integral Representations

The theory applies to functions $f(x)$ defined only on $0 < x < \infty$.

Definition. The Fourier Cosine Integral Representation

$$f(x) = \int_0^\infty A(w)\cos(wx)dw, \quad A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos(wt)dt$$

Definition. The Fourier Sine Integral Representation

$$f(x) = \int_0^\infty B(w)\sin(wx)dw, \quad B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin(wt)dt$$

Fourier Sine and Cosine Transforms

Definition. The Fourier Cosine and Sine Forward Transforms

$$FCT[f](w) = \frac{2}{\pi}\int_0^\infty f(t)\cos(wt)dt,$$

$$FST[f](w) = \frac{2}{\pi}\int_0^\infty f(t)\sin(wt)dt$$

Definition. The Fourier Cosine and Sine Inverse Transforms

$$f(x) = \int_0^\infty FCT[f](w)\cos(wx)dw,$$

$$f(x) = \int_0^\infty FST[f](w)\sin(wx)dw$$

Fourier Sine and Cosine Transform Properties

Theorem 1 (Properties)

- $FCT[f](w) = 2FT[f](w)$ for $w \geq 0$, provided f is even on $(-\infty, \infty)$.
- $FST[f](w) = 2FT[f](w)$ for $w \geq 0$, provided f is odd on $(-\infty, \infty)$.
- Both FCT and FST satisfy the Fourier transform's linearity property.
- $FCT[f'] = w\, FST[f] - \frac{2}{\pi} f(0)$, provided $\lim_{x \to \infty} f(x) = 0$.
- $FST[f'] = -w\, FCT[f]$, provided $\lim_{x \to \infty} f(x) = 0$.
- $FCT[xf(x)] = \frac{d}{dw} FST[f]$
- $FST[xf(x)] = -\frac{d}{dw} FCT[f]$