Name ____________________________

Math 3150 Problems
Chapter 7

Due date: See the internet due date. Problems are collected once a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one package per problem. Label each problem with its corresponding problem number, e.g., $\boxed{\text{Prob3.1-4}}$ or $\boxed{\text{Xc1.2-4}}$. Kindly label extra credit problems with label $\boxed{\textbf{Extra Credit}}$. You may attach this printed sheet to simplify your work.

Labeling. The label $\boxed{\text{Probx.y-z}}$ means the problem is for chapter $\boxed{\text{x}}$, section $\boxed{\text{y}}$, problem $\boxed{\text{z}}$. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., $\boxed{\text{Xc1.0-4}}$, $\boxed{\text{Xc1.1-2}}$.

Chapter 7: 7.1 – Fourier Integral Representation

Prob7.1-8a. (Fourier Integral Formulas)

Find functions $A(\omega)$, $B(\omega)$ for $f(x) = \begin{cases} 1, & 1 < |x| < 2, \\ 0 & \text{otherwise,} \end{cases}$ in the Fourier integral representation

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega.$$

Display all details, including computer algebra steps, in evaluating the integrals

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt.$$

Prob7.1-8b. (Fourier Integral Convergence)

Given $f(x) = \begin{cases} 1, & 1 < |x| < 2, \\ 0 & \text{otherwise,} \end{cases}$, report the values of x for which $f(x)$ equals its Fourier integral.

Prob7.1-19. (Fourier Integral and Integration Formulas)

Invent a function $f(x)$ such that the Fourier Integral Representation implies the formula

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega x)}{1+\omega^2}\,d\omega.$$

Chapter 7: 7.2-7.3 – Fourier Transform

Prob7.2-20. (Fourier Transform)

Let $f(x) = x$ for $|x| < 1$ and $f(x) = 0$ elsewhere. Find the fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Reference: See Asmar 1E problem 7.2-20 or Asmar 2E problem 7.2-39.

Prob7.2-18a. **(Fourier Transform Rules)**

State in detail the following operational rules for the Fourier Transform.

(a) Linearity.	(b) x-differentiation rule [parts].	(c) ω-differentiation rule.	(d) Convolution rule.
(e) x-axis shifting rule.	(f) ω-axis shifting rule.	(g) t-partial derivative rule.	

Reference: Asmar's text, *PDE and BVP*, section 7.2, Theorems 1,2,3,4,5 and problems 7.2-19, 7.2-20. Section 7.3 for item (g). See also appendix B in Asmar.

DETAILS. State a minimal rule, one which can easily generate all variants of the rule. For example, all ω-shifting rules arise from the single identity $\mathcal{F}[e^{iax}f(x)](\omega) = \mathcal{F}[f(x)](\omega - a)$.

Prob7.2-18b. **(Fourier Transform Table)**

Derive the table below, considered to be a suitable minimal table for using the Fourier transform on applied problems. In the table, we use the Fourier Transform notation

$$\mathcal{F}(f(x)) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

Define the **pulse function** by

$$\textbf{pulse}(x, a, b) = \begin{cases} 1 & a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{F}(\textbf{pulse}(x,a,b))(\omega)$	$=$	$\dfrac{e^{-i\omega a} - e^{-i\omega b}}{i\omega\sqrt{2\pi}}$
$\mathcal{F}\left(\frac{1}{a^2+x^2}\right)(\omega)$	$=$	$\dfrac{\pi e^{-a\lvert\omega\rvert}}{a\sqrt{2\pi}}$ $\quad (a > 0)$
$\mathcal{F}\left(e^{-a\lvert x\rvert}\right)(\omega)$	$=$	$\dfrac{1}{\sqrt{2\pi}} \dfrac{2a}{a^2+\omega^2}$ $\quad (a > 0)$
$\mathcal{F}\left(e^{-ax^2}\right)(\omega)$	$=$	$\dfrac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$ $\quad (a > 0)$

Reference: Asmar's text, *PDE and BVP*, section 7.2 and appendix B.

Prob7.2-31a. **(Fourier Transform Calculus)**

Use the Fourier transform rules and Fourier transform table to derive a formula for $\mathcal{F}(f(x))$.

(a) $f(x) = \cos(x)\, e^{-x^2}$	(b) $f(x) = \sin(x)\, e^{-\lvert x\rvert}$	(c) $f(x) = \cos(x)\, \dfrac{1}{1+x^2}$

DETAILS. The table entries are found from the ω-shifting rule, $\mathcal{F}[e^{iax}f(x)](\omega) = \mathcal{F}[f(x)](\omega - a)$, using identities that write $\cos(bx)$ and $\sin(bx)$ in terms of complex exponentials e^{ibx} and e^{-ibx}.

Xc7.2-31a. **(Fourier Transform Calculus)**

Use the Fourier transform rules and Fourier transform table to derive a formula for $\mathcal{F}(f(x))$.

(a) $f(x) = \cos(2x)\dfrac{1}{4+x^2}$	(b) $f(x) = \cos(x)\ \textbf{pulse}(x, -1, 1)$	(c) $f(x) = \sin(x)\, \textbf{pulse}(x, -1, 1)$

DETAILS. The same advice applies as in Problem 7.2-31a.

Prob7.2-45. **(Fourier Transform Convolution)**

(a) Find the Fourier transform of the convolution of $xe^{-x^2/2}$ and e^{-x^2}.

(b) Solve for $h(x)$ in the equation $\mathcal{F}(h(x)) = e^{-\omega^2} \dfrac{\sin \omega}{\omega}$, using the convolution theorem.

Xc7.2-47. **(Fourier Transform Convolution)**

Write a proof for the convolution identities

(a) $f * g = g * f$
(b) $f * (g * h) = (f * g) * h$

DETAILS. The convolution of f and g is defined by the equation $f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt$.

Reference: See Asmar 1E problem 7.2-47, identical to Asmar 2E problem 7.2-55.

Prob7.3-1. **(Fourier Transform Method Wave Equation)**

Solve the boundary value problem on $-\infty < x < \infty$, $t > 0$.

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t), \\ u(x,0) &= \frac{1}{1+x^2}, \\ u_t(x,0) &= 0. \end{aligned}$$

Prob7.3-4. **(Fourier Transform Method Heat Equation)**

Solve the boundary value problem on $-\infty < x < \infty$, $t > 0$.

$$\begin{aligned} u_t(x,t) &= \frac{1}{100} u_{xx}(x,t), \\ u(x,0) &= 100\,\mathbf{pulse}(x,-1,1). \end{aligned}$$

Xc7.3-17. **(Fourier Transform Method Infinite Beam)**

Solve the boundary value problem on $-\infty < x < \infty$, $t > 0$.

$$\begin{aligned} u_{tt}(x,t) &= \frac{1}{100} u_{xxxx}(x,t), \\ u(x,0) &= 100\,\mathbf{pulse}(x,-2,2). \end{aligned}$$

Chapter 7: 7.4-7.5 – Heat Kernel and Poisson Integral Formula

Xc7.4-2. **(Heat Kernel)**

Solve the boundary value problem on $-\infty < x < \infty$, $t > 0$.

$$\begin{aligned} u_t(x,t) &= \frac{1}{100} u_{xx}(x,t), \\ u(x,0) &= 100\,\mathbf{pulse}(x,-2,0) + 50\,\mathbf{pulse}(x,0,1). \end{aligned}$$

Xc7.4-6. **(Heat Kernel)**

Solve the boundary value problem on $-\infty < x < \infty$, $t > 0$.

$$\begin{aligned} u_t(x,t) &= \frac{1}{100} u_{xx}(x,t), \\ u(x,0) &= e^{-|x|}. \end{aligned}$$