Name ___________________________

Math 3150 Problems
Haberman Chapter H10, Fourier Transform

Due Date: Problems are collected on Wednesday.

Chapter H10: 10.4, 10.5 Fourier Transform

EXERCISES H10.4, Fourier Transform and the Heat Equation

Problem H10.4-2. (Heat Equation on $-\infty < x < \infty$, Limit Zero at Infinity)

For the heat equation,

$$u(x,t) = \int_{-\infty}^{\infty} F(w) e^{-iwx} e^{-kw^2 t} dw.$$

Show that $\lim_{x=\infty} u(x,t) = 0$ even though $\phi(x) = e^{-iwx}$ does not decay as $x \to \infty$. (Hint: Integrate by parts.)

Problem H*10.4-3. (Diffusion-Convection Equation)

(a) Solve the diffusion equation with convection:

$$u_t(x,t) = k u_{xx}(x,t) + c u_x(x,t), \quad -\infty < x < \infty, \quad t > 0,$$

subject to $u(x,0) = f(x)$.

[Hint: Use the convolution theorem and the shift theorem (see Exercise H10.4-5).]

(b) Consider the initial condition to be the Dirac unit impulse $\delta(t)$. Sketch the corresponding diffusion-convection solution $u(x,t)$ for various values of $t > 0$. Comment on the significance of the convection term $c u_x(x,t)$.

Problem H*10.4-5. (Diffusion Equation with Source $Q(x,t)$)

Consider the diffusion equation

$$u_t(x,t) = k u_{xx}(x,t) + Q(x,t), \quad -\infty < x < \infty, \quad t > 0,$$

with initial condition $u(x,0) = f(x)$.

(a) Show that a particular solution for the Fourier transform $U(w) = \mathbf{FT}[u(x,t)]$ is

$$U_1(w) = e^{-kw^2 t} \int_0^t Q_1(w,r) e^{kw^2 r} dr, \quad Q_1(w,t) = \mathbf{FT}[Q(x,t)].$$

(b) Determine U_1.

$*$(c) Solve for $u(x,t)$ (in the simplest form possible).

Answer: $u(x,t)$ is the Heat kernel solution $u_0(x,t)$ of the homogeneous problem plus the inverse Fourier transform of U_1, which is

$$u_1(x,t) = \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} Q(v,t) \sqrt{\frac{\pi}{k(t-r)}}\, e^{-\frac{(x-v)^2}{4k(t-r)}}\, dv\, dr.$$

Problem XC-H10.4-11. (Fourier Transform of a Product)

Derive an expression for the Fourier transform of the product $f(x)g(x)$.

Answer: $\mathbf{FT}[f(x)g(x)]$ is the convolution of $\mathbf{FT}[f(x)]$ with $\mathbf{FT}[g(x)]$.

EXERCISES H10.5, Fourier Sine and Cosine Transforms