Name ___________________________

Math 3150 Problems
Haberman Chapter H10, Fourier Transform

Due Date: Problems are collected on Wednesday.

Chapter H10: 10.2, 10.3, 10.4 Fourier Transform

EXERCISES H10.2

Problem H*10.2-1. (Heat Equation on $-\infty < x < \infty$, Coefficient Identities)

Determine complex $c(w)$ so that

$$u(x,t) = \int_{-\infty}^{\infty} c(w)e^{-iwx}e^{-kw^2t}dw$$

is equivalent to

$$u(x,t) = \int_{0}^{\infty} \left(A(w)\cos(wx) + B(w)\sin(wx))e^{-kw^2t}\right) dw$$

with real $A(w)$ and $B(w)$. Then show that $c(-w) = \bar{c}(w)$, $w > 0$, where the over-bar denotes the complex conjugate.

Problem H10.2-2. (Heat Equation, Complex Integrand)

If $c(-w) = \bar{c}(w)$ (see the preceding exercise), then show that $u(x,t)$ is real, where

$$u(x,t) = \int_{-\infty}^{\infty} c(w)e^{-iwx}e^{-kw^2t}dw$$

EXERCISES H10.3

Problem H10.3-1. (Linearity of the Fourier Transform)

Show that the Fourier transform is a linear operator; that is, show that
(a) $FT[c_1f(x) + c_2g(x)] = c_1FT[f(x)] + c_2FT[g(x)]$
(b) $FT[f(x)g(x)] \neq FT[f(x)]FT[g(x)]$

Problem H10.3-2. (Linearity of the Inverse Fourier Transform)

Show that the inverse Fourier transform is a linear operator; that is, show that
(a) $FT^1[c_1FT[f(x)] + c_2FT[g(x)]] = c_1f(x) + c_2g(x)$
(b) $FT^{-1}[F(w)G(w)] \neq f(x)g(x)$

Problem H10.3-3. (Complex Conjugate and Fourier Transform)

Let $F(w)$ be the Fourier transform of $f(x)$. Show that if $f(x)$ is real, then $F^*(w) = F(-w)$, where $*$ denotes the complex conjugate.

Problem XC-H10.3-4. (Transforms of Functions Depending on a Parameter α)

Show that $FT\left[\int f(x;\alpha)d\alpha\right] = \int F(w,\alpha)d\alpha$.

Problem H10.3-5. (Shift and the Fourier Transform)

If $F(w)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{iw\beta}F(w)$ is $f(x-\beta)$. This result is known as the **shift theorem** for Fourier transforms.

Problem H*10.3-6. (Transform of the Unit Pulse: Sinc Function, Rect Pulse)

If $f(x) = \begin{cases} 0 & |x| > a, \\ 1 & |x| < a, \end{cases}$ then determine the Fourier transform of $f(x)$.

Answer: $\dfrac{1}{\pi}\dfrac{\sin(aw)}{w}$, or $\dfrac{a}{\pi}\,\mathbf{sinc}(aw)$. The **sinc** function is a widely research function in numerical analysis, defined by $\mathbf{sinc}(u) = \dfrac{\sin(u)}{u}$.

Remark. A standard transform table may contain instead the function **rect**, a rectangular pulse of width 1 with value $\frac{1}{2}$ at $x = \pm\frac{1}{2}$.

[The answer is given in the table of Fourier transforms in H10, Section 4.4.]

Problem H*10.3-7. (Transform Table, Exponential Transform)

If $F(w) = e^{-|w|\alpha}$, $\alpha > 0$, then determine the inverse Fourier transform of $F(w)$.

Answer: $f(x) = FT^{-1}(F(w)) = \dfrac{2\alpha}{x^2 + \alpha^2}$.

[The answer is given in the table of Fourier transforms in H10, Section 4.4.]

Problem XC-H10.3-8. (Multiply by x and Differentiation of $F(w)$)

If $F(w)$ is the Fourier transform of $f(x)$, show that $-i\,dF/dw$ is the Fourier transform of $xf(x)$.

Problem XC-H10.3-9. (Textbook Details)

(a) Multiply (10.3.6) (assuming that $\gamma = 1$) by e^{-iwx} and integrate from $-L$ to L to show that

$$\int_{-L}^{L} F(w)e^{-iwx}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\frac{2\sin(L(u-x))}{u-x}du. \quad (10.3.13)$$

(b) Derive (10.3.7). For simplicity, assume that $f(x)$ is continuous. [Hints: Let $f(u) = f(x) + f(u) - f(x)$. Use the sine integral, $\int_0^\infty \frac{\sin s}{s}ds = \frac{\pi}{2}$ Integrate (10.3.13) by parts and then take the limit as $L \to \infty$.

Problem XC-H10.3-11. (Scaling)

(a) If $f(x)$ is a function with unit area, $\int_{-\infty}^{\infty} f(x)dx = 1$, show that the scaled and stretched function $(1/\alpha)f(x/\alpha)$ also has unit area.

(b) If $F(w)$ is the Fourier transform of $f(x)$, show that $F(\alpha w)$ is the Fourier transform of $(\alpha)f(x/\alpha)$.

(c) Show that part (b) implies that broadly spread functions have sharply peaked Fourier transforms near $w = 0$, and vice versa.

Problem XC-H10.3-13. (Cosine)

Evaluate $\int_0^\infty e^{-kw^2t}\cos(wx)dw$ in the following way. Determine $\partial I/\partial x$, and then integrate by parts.

Problem H10.3-14. (Gamma Function)

The gamma function r(x) is defined as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Show that

(a) $\Gamma(1) = 1$ (b) $\Gamma(x+1) = \Gamma(x)$

(c) $\Gamma(n+1) = n!$ (d) $\Gamma(1/2) = 2\int_0^\infty e^{-t^2}dt = \sqrt{\pi}$

(e) What is $\Gamma(3/2)$?

Problem XC-H10.3-15. **(Gamma Function Properties)**

(a) Using the definition of the gamma function in the previous Exercise, show that

$$\Gamma(x) = 2\int_0^\infty u^{2x-1}e^{-u^2}\,du.$$

(b) Using double integrals in polar coordinates, show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

[Hint: It is known from complex variables that $2\int_0^{\pi/2}(\tan\theta)^{2x-1}d\theta = \frac{\pi}{\sin(\pi z)}$.]

Problem XC-H*10.3-16. **(Gamma Function Identity)**

Evaluate $\int_0^\infty y^p e^{-ky^n}\,dy$ in terms of the gamma function (see Exercise 10.3.14).