Name ___________________________

Math 3150 Problems
Haberman Chapter H3

Due Date: Problems are collected on Wednesday.

Chapter H3: 3.4 Differentiation and Integration of Fourier Series

Problem H3.4-1. (Integration by Parts)

The integration-by-parts formula

$$\int_a^b u\,dv = uv|_x^b - \int_a^b v\,du$$

is known to be valid for functions $u(x)$ and $v(x)$ which are continuous and have continuous first derivatives. However, we will assume that $u, v, du/dx$, and dv/dx are continuous only for $a < x < c$ and $c < x < b$; we assume that all quantities may have a jump discontinuity at $x = c$.

$*$(a) Derive an expression for $\int_a^b u\,dv$ in terms of $\int_a^b v\,du$.

(b) Show that this reduces to the integration-by-parts formula if u and v are continuous across $x = c$. It is not necessary for du/dx and dv/dx to be continuous at $x = c$.

Problem H3.4-2. (Fourier Series Differentiation)

Suppose that $f(x)$ and df/dx are piecewise smooth. Prove that the Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous.

Problem H3.4-3. (Fourier Coefficients of f')

Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0, f(x_0-) = \alpha$ and $f(x_0+) = \beta$] and df/dx is piecewise smooth.

$*$(a) Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

Problem XC-H3.4-4. (Differentiation of Fourier Sine and Cosine Series)

Suppose that $f(x)$ and df/dx are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can only be differentiated term by term if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

Problem XC-H3.4-5. (Fourier Cosine Series by Differentiation of a Sine Series)

Determine the Fourier cosine series of $\sin(\pi x/L)$, using Fourier sine series formulas developed in the previous section.

Problem H3.4-6. (Series Differentiation Mistakes)

There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L).$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sin(n\pi x/L),$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos(n\pi x/L).$$

Next, we match coefficients in the first and third series, because they are both cosine series. Thus

$$A_0 = 0, \quad A_n = 0, \quad \text{Obviously wrong!}$$

By correcting the mistakes, you should be able to obtain A_0 and A_n without using the typical technique, that is, $A_n = (2/L)\int_0^L e^x \cos(n\pi x/L)dx$.

Problem XC-H3.4-7. **(Fourier Series with Extra Time Variable)**

Prove that the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to the parameter t if $\partial u/\partial t$ is piecewise smooth.

Problem H3.4-8. **(Insulated Rod Superposition Series Justification)**

Consider

$$u_t = ku_{xx}$$

subject to $u_x = 0$ at $x = 0$ and $x = L$, and $u(x,0) = f(x)$. Solve in the following way. Look for the solution as a Fourier cosine series. Assume that u and u_x are continuous and u_{xx} and u_t are piecewise smooth. Justify all differentiations of infinite series.

Problem H3.4-9. **(Ice-Pack Rod with Heat Source $q(x,t)$)**

Consider the heat equation with a known source $q(x,t)$:

$$u_t = ku_{xx} + q(x,t)$$

with $u(0,t) = 0$ and $u(L,t) = 0$. Assume that $q(x,t)$ (for each $t > 0$) is a piecewise smooth function of x. Also assume that u and u_x are continuous functions of x (for $t > 0$) and u_{xx} and u_t are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x/L).$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

Problem XC-H3.4-10. **(Insulated Rod with Heat Source $q(x,t)$)**

Modify the previous exercise if instead $u_x = 0$ at $x = 0$ and $x = L$.

Problem XC-H3.4-11. **(Ice-Pack Rod with Steady Heat Source $g(x)$)**

Consider the nonhomogeneous heat equation (with a steady heat source):

$$u_t = ku_{xx} + g(x).$$

Solve this equation with the initial condition $u(x,0) = f(x)$ and the boundary conditions $u(O,t) = 0$ and $u(L,t) = 0$. Assume that a continuous solution exists (with continuous derivatives). [Hints: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

Problem XC-H3.4-12. **(Insulated Rod with Explicit Heat Source $q(x,t)$)**

Solve the following nonhomogeneous problem:

$$u_t = ku_{xx} + e^{-t} + e^{-2t}\cos(3\pi x/L) \quad [\text{assume that } 2 \neq k(3\pi/L)^2]$$

subject to $u_x = 0$ at $x = 0$ and $x = L$, and $u(x,0) = f(x)$. Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

Problem XC-H3.4-13. **(Rod with Specified Endpoint Temperatures)**

Consider

$$u_t = ku_{xx}$$

subject to $u(0,t) = A(t)$, $u(L,t) = 0$, and $u(x,0) = g(x)$. Assume that $u(x,t)$ has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).