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Math 3150 Problems
Haberman Chapter H2

Due Date: Problems are collected on Wednesday.

Chapter H2: 2.4 Heat Equation, Worked Examples

Problem H2.4-1. (Heat BVP, Both Ends Insulated)

Solve the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $0 < x < L$, $t > 0$, subject to $u_x(0,t) = 0$ and $u_x(L,t) = 0$, $t > 0$.
(a) $u(x,0) = 0$ on $x < L/2$ and $u(x,0) = 1$ for $x > L/2$
(b) $u(x,0) = 6 + 4\cos(3\pi x/L)$
(c) $u(x,0) = -2\sin(\pi x/L)$
(d) $u(x,0) = -3\cos(8\pi x/L)$
Reference. Haberman H2.4.

Problem H2.4-2. (Heat BVP, One End Insulated, One End Ice-Pack)

Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

on $0 < x < L$, $t > 0$, with $u_x(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$.
For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

Problem H2.4-3. (Eigenvalue Problem, Perfect Thermal Contact)

Solve the eigenvalue problem $X'' + \lambda X = 0$ subject to $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$.

Problem H2.4-4. (Eigenvalue Problem, Insulated Ends)

Explicitly show that there are no negative eigenvalues for $X'' + \lambda X = 0$ subject to $X'(0) = 0$ and $X'(L) = 0$.

Problem XC-H2.4-5. (Heat Equation Derivation, Thin Wire)

This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

if the temperature does not depend on r and if the wire is very thin.

Problem H2.4-6a. (Equilibrium Temperature, Thin Circular Ring)

Determine the equilibrium temperature $u = \frac{1}{2L}\int_{-L}^{L} f(x)dx$ for the thin circular ring, directly from the equilibrium problem $u''(x) = 0$, $u(-L) = u(L)$, $u'(-L) = u'(L)$ (solution $u = u_0 =$ constant), by completing these steps for evaluating u_0.

1. The equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ is integrated on $x = -L$ to $x = L$. Obtain the identity

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)dx = k\int_{-L}^{L} u_{xx}(x,t)dx.$$

2. Evaluate the right side of the above equation using the boundary condition $u_x(-L,t) = u_x(L,t) = 0$. You should get zero. Assuming c, ρ, A are constants, conclude from the left side that for some c_1

$$\int_{-L}^{L} u(x,t)dx = \text{ constant } = c_1$$

3. By taking limits as $t \to \infty$, the value u_0 of the equilibrium temperature can replace $u(x,t)$ in the preceding integral to obtain $2Lu_0 = c_1$. By setting $t = 0$ and using $u(x,0) = f(x)$, the preceding integral gives

$$\int_{-L}^{L} f(x)dx = c_1.$$

Show all details here and conclude that the equilibrium temperature is $u_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$.
Reference: H2.4 equations (25), (26), (27) for the thin insulated circular ring..

Problem XC-H2.4-6b. (Equilibrium Temperature, Thin Circular Ring)

Determine the equilibrium temperature distribution for the thin circular ring by directly computing the limit as t approaches infinity of the answer to the time-dependent problem.
Reference: H2.4 equations (38) and (43).

Problem XC-H2.4-7. (Laplace's Equation, Circle of Radius a)

Solve Laplace's equation inside a circle of radius a,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition $u(a,\theta) = f(\theta)$.