

Name \_\_\_\_\_

**Math 3150 Problems**  
**Haberman Chapter H1**

**Due Date:** Problems are collected on Wednesday.

**Submitted work.** Please submit one stapled package per problem set. Label each problem with its corresponding problem number, e.g., Problem H3.1-4. Labels like Problem XC-H1.2-4 are extra credit problems. Attach this printed sheet to simplify your work.

**Labels Explained.** The label Problem Hx.y-z means the problem is for Haberman 4E or 5E, chapter x, section y, problem z. Label Problem H2.0-3 is background problem 3 in Chapter 2 (take  $y = 0$  in label Problem Hx.y-z). If not a background problem, then the problem number should match a corresponding problem in the textbook. The chapters corresponding to 1,2,3,4,10 in Haberman 4E or 5E appear in the hybrid textbook Edwards-Penney-Haberman as chapters 12,13,14,15,16.

**Required Background.** Only calculus I and II plus engineering differential equations are assumed. If you cannot understand a problem, then read the references and return to the problem afterwards. Your job is to understand the Heat Equation.

## Chapter H1: 1.1–1.2 – Introduction, Heat Equation

### Problem H1.0-1. (Partial Derivatives)

Compute the partial derivatives.

$\frac{\partial}{\partial x}(1 + t + x^2)$	$\frac{\partial}{\partial x}(1 + xt + (tx)^2)$	$\frac{\partial}{\partial x}(\sin xt + t^2 + tx^2)$
$\frac{\partial}{\partial t}(\sinh t \sin 2x)$	$\frac{\partial}{\partial t}(e^{2t+x} \sin(t+x))$	$\frac{\partial}{\partial t}(e^{3t+2x}(e^t \sin x + e^x \cos t))$

### Problem H1.0-2. (Jacobian)

Find the Jacobian matrix  $J$  and the Jacobian determinant  $|J|$  for the transformation  $\mathbf{F}(x, y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ , where  $f_1(x, y) = \cos(2y) \sin(3x)$ ,  $f_2(x, y) = e^y \sin(y+x)$ .

**Background:**  $J$  is the augmented  $2 \times 2$  matrix of column vectors  $\partial_x \mathbf{F}(x, y)$  and  $\partial_y \mathbf{F}(x, y)$ . For a  $2 \times 2$  matrix  $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ , the determinant is defined by  $|A| = ad - bc$ .

### Problem H1.0-3. (Directional Derivative)

Compute the gradient vector  $\mathbf{grad}(f)$  and the directional derivative of  $f$  in the direction of  $\mathbf{v} = 2\vec{i} - \vec{j}$  at  $x = y = 1$ .

**Background:**  $\mathbf{grad}(f) = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}$ ,  $DD(f) = \mathbf{grad}(f) \cdot \mathbf{v}/|\mathbf{v}|$ .

$f = x + y + 1$	$f = x^2 + y^2$	$f = ye^{x-1}$
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### Problem H1.0-4. (First Order Linear ODE)

Solve the following first order ODE problems.

$y' - y = 0$	$y' + 2y = 0$	$y' + 2y = 4$
$xy' + y = x + 1$	$y' = xy$	$y' = y^2$

References: Edwards-Penney, Chapter 1. Appendix A of Asmar, any edition. The first five are linear, the last separable.

### Problem H1.0-5. (Wronskian)

Compute the Wronskian determinant of the given functions at  $x = 0$ .

$1, x$	$1, e^x$	$\cos x, \sin x$
$e^x, e^{2x}, e^{3x}$	$1, \cosh x, \sinh x$	$1, \ln  1 + x , x \ln  1 + x $

**References:** Edwards-Penney, *DE and LA*, Chapter 5. Appendix A of Asmar, any edition.

**Background:** The Wronskian of  $f, g$  is the determinant  $\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ . For functions  $f_1, \dots, f_n$ , the Wronskian is the determinant of the  $n \times n$  matrix formed by taking row 1 to be  $f_1, \dots, f_n$  and then successive rows are the successive derivatives of row 1.

### Problem H1.0-6. (Newton's Laws)

(a) State Newton's second law  $F = ma$  in calculus terms for a motion of a mass  $m$  in motion along a line with position  $x(t)$  at time  $t$ .

(b) Use the statement in (a) to derive the spring-mass undamped free vibration model  $mx''(t) + kx(t) = 0$ . Assume no external forces and a Hooke's restoring force on the spring of magnitude  $kx(t)$ . Symbol  $x(t)$  is the signed distance from the center of mass to equilibrium position  $x = 0$ . Symbol  $m$  is the mass, symbol  $k$  is the Hooke's constant.

**References:** Edwards-Penney Chapter 5. The ODE in (b) is called the **harmonic oscillator**. Serway and Vuille, *College Physics*, 10E.

### Problem H1.0-7. (PDE Answer Check: Advection Equation)

Show the details of an answer check for the solution  $u(x, t) = e^{-(x-t)^2}$  of the initial value problem  $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ ,  $u(0, t) = e^{-t^2}$ .

Draw four (4) pulse figures for the waveform  $u$ , for  $x = 0, 1, 2, 3$ . Each image is a 2-dimensional plot in the  $tu$ -plane with  $t$ -range  $-2$  to  $5$ , called a **snapshot** or a **frame** (from *movie frame*).

**References:** The PDE is called the *advection equation*. The term **advection** means that particles are simply carried by the bulk motion of the fluid. It is a special case of the advection-dispersion-reaction equation

$$\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + D_L \frac{\partial^2 C}{\partial x^2} - \frac{\partial q}{\partial t}.$$

The terms on the right are advection, dispersion and reaction. A historical reference (1999) for US Geological Survey applications, which fully explains the symbols in the above PDE, can be found [Here](http://pubs.er.usgs.gov/publication/wri994259). A more complete discussion of USGS applications is found at <http://pubs.er.usgs.gov/publication/wri994259>.

Hydraulics in civil engineering uses advection to model transport of a chemical (e.g., a pollutant, a lump of ink) in a stream of water flowing through a thin tube at constant speed  $c$ .

In a simple model, the ink moves as a lump or pulse, without dispersion. If a bottle of ink is dumped in a river and  $u(x, t)$  is the downstream ink concentration of the moving lump of ink, then dispersion becomes important.

### Problem H1.2-1. (Minus Sign)

Explain the minus sign in the conservation law  $\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$ .

**Example H1.2-1a.** Explain the minus sign in conservation law  $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$ .

**Solution.** The equation to be explained is  $de/dt = -(d/dx)\phi$ . To make sense of the equation we replace derivatives by Newton difference quotients, as follows:

$$de/dt \approx \frac{e(x, t+h) - e(x, t)}{h}$$

$$(d/dx)\phi(x, t) \approx \frac{\phi(x+k, t) - \phi(x, t)}{k}.$$

Suppose the thermal energy density  $e(x, t)$  increases in the time interval  $t$  to  $t+h$ , e.g.,  $de/dt > 0$ . Then the difference quotient for  $e(x, t)$  is positive. On the other side of the equation, the amount of heat energy per unit time flowing to

the right had to decrease, that is more heat had to be trapped in the rod section in order to increase the heat density  $e(x, t)$ . So the right side of the equation is

$$-(d/dx)\phi = (-1)(\phi(x+k, t) - \phi(x, t)) = (-)(-) = +$$

which accounts for the minus sign.

### Problem H1.2-7. (Heat Equation Derivation)

Conservation of thermal energy for any segment of a one-dimensional rod  $a < x < b$  satisfies the equation

$$\frac{d}{dt} \int_a^b e(x, t) dx = \phi(a, t) - \phi(b, t) + \int_a^b Q dx$$

By using the fundamental theorem of calculus,  $(d/db) \int_a^b f(x) dx = f(b)$ , derive the heat equation

$$c\rho \frac{du}{dt} = \frac{d}{dx} \left( K_0 \frac{du}{dx} \right) + Q.$$

**References.** Haberman H1.2 and Asmar 3.5, Appendix.

### Problem H1.2-8. (Total Thermal Energy)

If  $u(x, t)$  is known, then give an expression for the time-dependent (because energy escapes at the ends) total thermal energy  $\int_0^L A(x)e(x, t) dx$  contained in a rod  $x = 0$  to  $x = L$ .

Validate the answer using a uniform rod of length  $L$  and cross-sectional area  $A$ , held at steady-state temperature  $u = u_0$ .

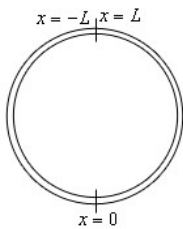
**Reference:** Serway and Vuille, College Physics, Chapter 11.

## Chapter H1: 1.3–1.4 – Heat Equation

### Problem H1.3-2. (Perfect Thermal Contact)

Two one-dimensional laterally insulated rods of different materials joined at  $x = x_0$  are said to be in perfect thermal contact if both the temperature  $u(x, t)$  and the heat flux  $\phi(x, t)$  at location  $x$  and time  $t$  varies continuously across the juncture  $x = x_0$ .

- (a) The continuous temperature equation at the juncture  $x = x_0$  is  $u(x_0-, t) = u(x_0+, t)$ . Explain the symbols. Illustrate the physical situation with a diagram.
- (b) Heat energy flowing out of one rod material at  $x = x_0$  flows into the other with no heat loss, which is the heat flux equation  $\phi(x_0+, t) = \phi(x_0-, t)$ . Apply Fourier's Law to re-write this condition as  $K_0(x_0+) \frac{\partial u}{\partial x}(x_0+, t) = K_0(x_0-) \frac{\partial u}{\partial x}(x_0-, t)$ .



(c)

A bar of length  $2L$  is bent into a solid ring. It then has no physical endpoints for energy to escape. Assuming perfect thermal contact will lead to the boundary conditions. Give the assumptions and the details for the formulation of the corresponding periodic boundary value problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = f(x), \quad u(-L, t) = u(L, t), \quad \frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t).$$

Don't solve the boundary value problem, just explain the equations.

**References:** Haberman's text H1.2, heat flux, thermal energy and Fourier's Law. See also the 1959 textbook reference in [http://en.wikipedia.org/wiki/Perfect\\_thermal\\_contact](http://en.wikipedia.org/wiki/Perfect_thermal_contact). Applications for thermal contact conductance include heat sinks, internal combustion engines, space vehicles.

There is an equation for heat flux between two solid bodies  $A$  and  $B$ :  $q = \frac{T_1 - T_3}{\Delta x_{-A}/(k_A A) + 1/(h_c A) + \Delta x_B/(k_B A)}$ . Details are in [http://en.wikipedia.org/wiki/Thermal\\_contact\\_conductance](http://en.wikipedia.org/wiki/Thermal_contact_conductance) and also in Holman, J. P. (2009) *Heat Transfer, 10E*, McGraw-Hill Series in Mechanical Engineering, ISBN-10: 0073529362.

### Problem H1.4-1. (Equilibrium Temperature in a Rod)

Determine the equilibrium temperature distribution for a one-dimensional rod  $c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q$  with initial temperature  $u(x, 0) = f(x)$ , constant thermal properties and the following source and boundary conditions:

- (a)  $Q = 0, u(0) = 0, u(L) = T$
- (b)  $Q = 0, u(0) = T, u'(L) = \alpha$
- (c)  $Q/K_0 = x^2, u(0) = T, u'(L) = 0$
- (d)  $Q = 0, u'(0) - [u(0) - T] = 0, u'(L) = \alpha$

### Problem H1.4-2. (Heat Energy in a Rod)

Determine the heat energy generated per unit time inside the entire rod, from the equilibrium temperature distribution for the model  $c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + K_0x$  with boundary conditions zero at  $x = 0$  and  $x = L$ .

### Problem XC-H1.4-3. (Bi-Metal Rod)

Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in **perfect thermal contact** at  $x = 1$ . For  $x = 0$  to  $x = 1$ , there is one material ( $c\rho = 1, K_0 = 1$ ) with a constant source ( $Q = 1$ ), whereas for the other  $x = 1$  to  $x = 2$  ( $c\rho = 2, K_0 = 2$ ) there are no sources ( $Q = 0$ ) with  $u(1) = 0$  and  $u(2) = 0$ . Reference: Haberman problem H1.3-2 and sections H1.2, H1.3.

### Problem H1.4-7. (Equilibrium Temperature)

Find the values of  $\beta$  for which there exists an equilibrium temperature distribution for the model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, \quad u(x, 0) = f(x), \quad \frac{\partial u}{\partial x}(0, t) = 1, \quad \frac{\partial u}{\partial x}(L, t) = \beta.$$

References: Haberman H1.4, equilibrium temperature. The physical interpretation of the model is a laterally insulated uniform rod with uniform internal heating along its length, with heat flux given at  $x = 0$  and  $x = L$  as 1 and  $\beta$ .

### Problem XC-H1.4-10. (Total Thermal Energy in a Rod)

Calculate the total thermal energy at time  $t$  in the one-dimensional rod with model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4, \quad u(x, 0) = f(x), \quad \frac{\partial u}{\partial x}(0, t) = 5, \quad \frac{\partial u}{\partial x}(L, t) = 6.$$

## Chapter 1: H1.5 – Heat Equation in Dimensions 2 and 3

### Problem H1.5-2. (Fourier's Heat Conduction Law, Convection and Advection)

The transfer of heat from a torched end of a welding rod to the far end is modeled by Fourier's Law of heat conduction,  $\phi = -K_0 \mathbf{grad}(u)$ . Water boiling in a pasta cooking pot illustrates **convection**, the movement of molecule aggregates in response to heat. Silt in a river illustrates **advection**, the movement in a fluid of suspended particles or dissolved material.

Suppose  $R$  is a region in space containing a material. A velocity field  $\vec{V}$  defines molecule movement: molecule aggregates in  $R$  move in direction  $\vec{V}$  with speed  $|\vec{V}|$ . Advection is assumed zero, due to no particulates.

Variables  $c, \rho, u, \vec{\phi}$  are functions of location  $(x, y, z)$  in region  $R$ , with  $u, \vec{\phi}$  also dependent on time  $t$ . Symbol  $c$  is the specific heat, the heat energy needed to add one degree of temperature to a unit mass of the material. Symbol  $\rho$  is material mass per unit volume. Symbol  $u$  is temperature. Vector symbol  $\vec{\phi}$  is heat flux, which satisfies Fourier's Law

$$\vec{\phi} = -K_0 \mathbf{grad}(u) \approx -(\text{thermal conductivity})(\text{temperature difference}).$$

Explain in fewest words or equations the following statements.

- (a) Thermal energy  $\vec{e}_1(x, y, z, t)$  is given by  $e_1 = c\rho u$ . Then heat flux from convection is  $e_1 \vec{V}$ .

- (b) Heat flux from conduction is given by Fourier's Law,  $-K_0 \mathbf{grad}(u)$ .
- (c) The total heat flux from conduction and convection is  $\vec{\phi} = -K_0 \mathbf{grad}(u) + c\rho u \vec{V}$ .
- (d) Assume symbols  $c, \rho, K_0$  are constants and  $Q = 0$ . Substitute the heat flux into the conservation law  $c\rho \frac{\partial u}{\partial t} = -\mathbf{Div}(\vec{\phi}) + Q$  to obtain for  $k = K_0/(c\rho)$  the partial differential equation  $\frac{\partial u}{\partial t} = k \Delta(u) - \vec{V} \cdot \mathbf{grad}(u) - u \mathbf{Div}(\vec{V})$ .

References: Haberman H1.5. In the case of an incompressible flow, the equation  $\mathbf{Div}(\vec{V}) = 0$  is valid, then (d) reduces to the equation  $\frac{\partial u}{\partial t} = k \Delta(u) - \vec{V} \cdot \mathbf{grad}(u)$ , which is the classical equation for heat conduction with convection, ignoring advection. Compare this equation to the chemical advection-dispersion-reaction equation  $\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + D_L \frac{\partial^2 C}{\partial x^2} - \frac{\partial q}{\partial t}$ .

**Problem XC-H1.5-14. (Isotherms and Insulated Boundary)**

Isotherms are lines of constant temperature. Show that isotherms are perpendicular to any part of the boundary that is insulated.

**Problem XC-H1.5-15. (Heat Equation 3D Derivation)**

Derive the heat equation in three dimensions assuming constant thermal properties and no sources.