Sample Exam 2
Solutions

Fourier
series
problem 1

1. **(Periodic Functions)**

(a) [30%] Find the period of $f(x) = \sin(x)\cos(2x) + \sin(2x)\cos(x)$.

(b) [40%] Let $p = 5$. If $f(x)$ is the odd $2p$-periodic extension to $(-\infty, \infty)$ of the function $f_0(x) = 100x\, e^{10x}$ on $0 \le x \le p$, then find $f(11.3)$. The answer is not to be simplified or evaluated to a decimal.

(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \quad \ln|2+\sin(x)| \quad \sin(2x)\cos(x) \quad \frac{1+\sin(x)}{2+\cos(x)}$$

Answer:

(a) $f(x) = \sin(x + 2x)$ by a trig identity. Then period $= 2\pi/3$.
(b) $f(11.3) = f(11.3 - p - p) = f(1.3) = f_0(1.3) = 130e^{13}$.
(c) All are periodic of period 2π, satisfying $f(x + 2\pi) = f(x)$. The first is even and the third is odd.

2 **(Fourier Series)**
Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's phenomenon?

Answer:

(a) Because $f_0(x)$ is even, then $f(x)$ is even. Then the coefficient of $\sin(67\pi x)$ is zero, without computation, because all sine terms in the Fourier series of f have zero coefficient. The coefficient of $\cos(n\pi x/2)$ for $n > 0$ is given by the formula

$$a_n = \frac{1}{2}\int_{-2}^{2} f_0(x)\cos(n\pi x/2)dx = \int_0^2 x\cos(n\pi x/2)dx.$$

For $\cos(2\pi x)$, we select $n\pi x/2 = 2\pi x$, or index $n = 4$.

(b) There are no jump discontinuities, f is continous, so no Gibbs overshoot.

3 **(Cosine and Sine Series)**
Find the first nonzero term in the sine series expansion of $f(x)$, formed as the odd 2π-periodic extension of the function $\sin(x)\cos(x)$ on $0 < x < \pi$. Leave the Fourier coefficient in integral form, unevaluated, unless you can compute the value in a minute or two.

Answer:

Because $\sin(x)\cos(x) = (1/2)\sin(2x)$ is odd and 2π-periodic, this is the Fourier series of f. This term is for coefficient b_2, so $b_2 = 1/2$ is the first nonzero Fourier coefficient. The first nonzero term is $(1/2)\sin(2x)$.

4 **(Convergence of Fourier Series)**

(a) [30%] Dirichlet's kernel formula can be used to evaluate the sum $\cos(2x) + \cos(4x) + \cos(6x) + \cos(8x)$. Report its value according to that formula.

(b) [40%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period $2p$. State the theorem for this special case, by translating the results when f is smooth and the interval $-\pi \le x \le \pi$ is replaced by $-p \le x \le p$.

(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's overshoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.

Answer:

(a) $\dfrac{1}{2} + \cos(x) + \cdots + \cos(nx) = \dfrac{\sin(nx + x/2)}{2\sin(x/2)}$ is used with x replaced by $2x$ and $n = 4$ to obtain the answer $0.5\sin(8x + x)/\sin(x) - 0.5$.
(b) Let f be a p-periodic smooth function on $(-\infty, \infty)$. Then for all values of x,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/p) + b_n\sin(n\pi x/p)),$$

where the Fourier coefficients a_0, a_n, b_n are given by the Euler formulas:

problem 4
Fourier
Series

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx, \quad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(n\pi x/p)dx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(n\pi x/p)dx.$$

(c) Any 2-periodic continuous function f will work, if we alter the values of f at the desired points to produce a jump discontinuity. For example, define $f(x) = \sin(\pi x)$ except at the points $\pm 2n$, where $f(x) = 2$ ($f(2n) = 2$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$).

From
S2013
final Exam
3150

5. (Fourier Series)

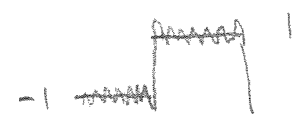
(a) [30%] Find and display the nonzero terms in the Fourier series expansion of $f(x)$, formed as the even 2π-periodic extension of the function $f_0(x) = \sin^2(x) + 4\cos(2x)$ on $0 < x < \pi$.

(b) [50%] Compute the Fourier sine series coefficients b_n for the function $g(x)$, defined as the period 2 odd extension of the function $g_0(x) = 1$ on $0 \le x \le 1$. Draw a representative graph for the partial Fourier sum for five terms of the infinite series.

(c) [20%] Define $h_0(x) = \begin{cases} \sin(2x) & 0 \le x < \pi, \\ x - \pi & \pi \le x \le 2\pi, \end{cases}$ and let $f(x)$ be the 4π odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $f(-5.25\pi) + f(1.5\pi)$.

5ⓐ $f(x) = \sin^2(x) + 4\cos(2x) = \frac{1-\cos 2x}{2} + 4\cos(2x) = \frac{1}{2} + \frac{7}{2}\cos(2x)$
It is even, So equals its periodic even extension.
ANSWER: $a_0 = \frac{1}{2}$, $a_2 = \frac{7}{2}$, all others are zero

5ⓑ Since g is odd, then all $a_n = 0$. Compute $b_n = \frac{1}{1}\int_{-1}^{1} g(x)\sin\left(\frac{n\pi x}{1}\right)dx$
$b_n = \frac{4}{n\pi}$ for n odd, others zero.



5 terms graphic

5ⓒ
$f(-5.25\pi) = f(4\pi - 5.25\pi)$
$= f(-1.25\pi)$
$= f(1.25\pi)$ odd f
$= 1.25\pi - \pi$ by $h_0(x)$ Definition
$f(1.5\pi) = 1.5\pi - \pi$ "
Sum $= 2.75\pi - 2\pi = \pi/4$

wave equation problem 1

1. (Vibration of a Finite String) The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= -11\sin(5\pi x). \end{aligned}$$

Answer:

Because the wave initial shape is zero, then the only normal modes are sine times sine. The initial wave velocity is already a Fourier series, using orthogonal set $\{\sin(n\pi x/2)\}_{n=1}^{\infty}$. The 1-term Fourier series $-11\sin(5\pi x)$ can be modified into a solution by inserting the missing sine factor present in the corresponding normal mode. Then $u(x,t) = -11\sin(5\pi x)\sin(5\pi ct)/(5\pi)$. We check it is a solution.

KEY

Partial Differential Equations 3150

Midterm Exam 2

Exam Date: Monday, 22 April 2013

Instructions: This exam is timed for 50 minutes. You will be given extra time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

2. **(CH3. Finite String: Fourier Series Solution)**

(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x,t) &= c^2 u_{xx}(x,t), & 0<x<L, & t>0, \\ u(0,t) &= 0, & & t>0, \\ u(L,t) &= 0, & & t>0, \\ u(x,0) &= f(x), & 0<x<L, & \\ u_t(x,0) &= g(x), & 0<x<L. & \end{cases}$$

Symbols f and g should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display an explicit formula for the Fourier coefficients which contains the symbols L, $f(x)$, $g(x)$.

(a) Normal modes: $\sin(n\pi x/L)\cos(n\pi ct/L)$, $\sin(n\pi x/L)\sin(n\pi ct/L)$

ans 1 $\longrightarrow$ $u(x,t)$ = superposition of the normal modes

$$= \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)\cos(n\pi ct/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\sin(n\pi ct/L)$$

(b) $f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$

ans 2 $\longrightarrow$ $a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$ by $\perp$ relations for $\{\sin(n\pi x/L)\}_{n=1}^{\infty}$

$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin(n\pi x/L)$

ans 3 $\longrightarrow$ $b_n = \frac{L}{n\pi c}\,\frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)\,dx$ by $\perp$ relations

Use this page to start your solution. Attach extra pages as needed, then staple.

3. (CH4. Rectangular Membrane)
Consider the general membrane problem

100

$$\begin{cases} u_{tt}(x,y,t) &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a,\quad 0<y<b,\quad t>0,\\ u(x,y,t) &= 0 & \text{on the boundary,}\\ u(x,y,0) &= f(x,y), & 0<x<a,\quad 0<y<b,\\ u_t(x,y,0) &= g(x,y), & 0<x<a,\quad 0<y<b. \end{cases}$$

Solve the problem for $a=b=c=1$, $f(x,y)=1$, $g(x,y)=0$. Expected are displays for the normal modes, a superposition formula for $u(x,y,t)$, and explicit numerical values for the generalized Fourier coefficients.

The solution is a superposition of the normal modes obtained from separation of variables as:

$$\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

with $\lambda_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right]$$

where the Fourier coefficients are:

$$B_{mn}=4\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$\lambda_{mn}B^*_{mn}=4\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

turn →
(answer continued on next page)

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'roblem 3 continued

plugging in:

$a = b = c = 1 \qquad f(x,y) = 1 \qquad g(x,y) = 0$

normal modes:

$$\sin(m\pi x)\sin(n\pi y)\left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]$$

$$\lambda_{mn} = \pi\sqrt{m^2+n^2}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(m\pi x)\sin(n\pi y)\left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]$$

Where the Fourier coefficient are:

$$B_{mn} = 4\int_0^1\int_0^1 \sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$= 4\int_0^1 -\cos(m\pi x)\sin(n\pi y)\left(\frac{1}{m\pi}\right)\Big|_0^1 dy$$

$$= 4\int_0^1 \left[-\cos(m\pi)\sin(n\pi y)\left(\frac{1}{m\pi}\right) - \sin(n\pi y)\left(\frac{1}{m\pi}\right)\right]dy$$

$$= 4\left[\cos(m\pi)\cos(n\pi y)\left(\frac{1}{mn\pi^2}\right) + \cos(n\pi y)\left(\frac{1}{mn\pi^2}\right)\right]\Big|_0^1$$

$$B_{mn} = 4\left[\cos(m\pi)\cos(n\pi)\left(\frac{1}{mn\pi^2}\right) + \cos(n\pi)\left(\frac{1}{mn\pi^2}\right) - \cos(m\pi)\left(\frac{1}{mn\pi^2}\right) - \left(\frac{1}{mn\pi^2}\right)\right]$$

$$B_{mn} = \frac{4}{mn\pi^2}\left[\cos(m\pi)\cos(n\pi) + \cos(n\pi) - \cos(m\pi) - 1\right]$$

turn $\rightarrow$

(answer continued on next page)

Problem 3 continued

$$\lambda_{mn} B^{*}_{mn} = 4\int_0^1\int_0^1 0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$= 4\int_0^1\int_0^1 0\,dx\,dy = 0$$

$$\boxed{B^{*}_{mn} = 0}$$

4. (Finite String: Fourier Series Solution)

(a) [50%] Display the series formula, **complete with derivation details**, for the solution $u(x,t)$ of the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0 < x < 2, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(2,t) &= 0, & & t > 0, \\ u(x,0) &= f(x), & 0 < x < 2, & \\ u_t(x,0) &= g(x), & 0 < x < 2. & \end{cases}$$

Symbols f and g should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display explicit formulas for the Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

(c) [25%] Evaluate the Fourier coefficients when $f(x) = 100$ and $g(x) = 0$.

Duplicate of problem 2, but evaluate coefficients.

See 3150 final exam, S 2013, http://math.utah.edu/~gustafso/s2013/

Fourier Transform

1. **(Fourier Transform Theory)**
   (a) [40%] Define Haberman's Fourier transform pair. Give an example of $f(x)$ and $F(w)$ which satisfy these equations.

   (b) [60%] The heat equation on the line $-\infty < x < \infty$ can be solved by Fourier transform methods. Outline the method, called Fourier's Method, for the example

$$u_t = 4u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x).$$

2. **(Fourier's Method)**
   Use the Heat kernel, the convolution theorem and the shift theorem to solve the diffusion-convection equation

$$u_t(x,t) = ku_{xx}(x,t) + cu_x(x,t), \quad t > 0, \quad -\infty < x < \infty, \quad u(x,0) = f(x).$$

   Answer: $u(x,t) = \dfrac{1}{\sqrt{4\pi kt}} \displaystyle\int_{-\infty}^{\infty} f(v) e^{\frac{-(x+ct-v)^2}{4kt}} dv$

3. **(Heat Equation and Gauss' Heat Kernel)**
   Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \frac{1}{4} u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

   Hint: Use the heat kernel $g_t(x,v) = \dfrac{1}{\sqrt{kt}} e^{-\frac{(x-v)^2}{4kt}}$, the error function $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$,