Partial Differential Equations 3150

Sample Midterm Exam 2
Exam Date: Wednesday, 9 April 2014

Instructions: This exam is timed for 50 minutes. Up to 60 minutes is possible. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

The actual exam will have three (3) problems, one selected from each main topic below. A problem may have several parts.

Fourier Series

1. (Periodic Functions)

(a) [30%] Find the period of $f(x) = \sin(x)\cos(2x) + \sin(2x)\cos(x)$.

(b) [40%] Let $p = 5$. If $f(x)$ is the odd $2p$-periodic extension to $(-\infty, \infty)$ of the function $f_0(x) = 100x\,e^{10x}$ on $0 \le x \le p$, then find $f(11.3)$. The answer is not to be simplified or evaluated to a decimal.

(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \quad \ln|2 + \sin(x)| \quad \sin(2x)\cos(x) \quad \frac{1 + \sin(x)}{2 + \cos(x)}$$

2. (Fourier Series)
Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's phenomenon?

3. (Cosine and Sine Series)
Find the first nonzero term in the sine series expansion of $f(x)$, formed as the odd 2π-periodic extension of the function $\sin(x)\cos(x)$ on $0 < x < \pi$. Leave the Fourier coefficient in integral form, unevaluated, unless you can compute the value in a minute or two.

4. (Convergence of Fourier Series)

(a) [30%] Dirichlet's kernel formula can be used to evaluate the sum $\cos(2x) + \cos(4x) + \cos(6x) + \cos(8x)$. Report its value according to that formula.

(b) [40%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period p. State the theorem for this special case, by translating the results when f is smooth and the interval $-\pi \le x \le \pi$ is replaced by $-p \le x \le p$.

(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's overshoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.

5. (Fourier Series)

(a) [30%] Find and display the nonzero terms in the Fourier series expansion of $f(x)$, formed as the even 2π-periodic extension of the function $f_0(x) = \sin^2(x) + 4\cos(2x)$ on $0 < x < \pi$.

(b) [50%] Compute the Fourier sine series coefficients b_n for the function $g(x)$, defined as the period 2 odd extension of the function $g_0(x) = 1$ on $0 \le x \le 1$. Draw a representative graph for the partial Fourier sum for five terms of the infinite series.

(c) [20%] Define $h_0(x) = \begin{cases} \sin(2x) & 0 \le x < \pi, \\ x - \pi & \pi \le x \le 2\pi, \end{cases}$ and let $h(x)$ be the 4π odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $h(-5.25\pi) + h(1.5\pi)$.

Wave Equation: Finite String, Membrane

1. (Vibration of a Finite String)
The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= -11\sin(5\pi x). \end{aligned}$$

2. (Finite String: Fourier Series Solution)
(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & 0 < x < L, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L, \\ u_t(x,0) = g(x), & 0 < x < L. \end{cases}$$

Symbols f and g should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display explicit formulas for the Fourier coefficients, containing the symbols L, $f(x)$, $g(x)$.

3. (Rectangular Membrane)
Consider the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a, \quad 0<y<b, \quad t>0, \\ u(x,y,t) &= 0 & \text{on the boundary}, \\ u(x,y,0) &= f(x,y), & 0<x<a, \quad 0<y<b, \\ u_t(x,y,0) &= g(x,y), & 0<x<a, \quad 0<y<b. \end{cases}$$

Solve the problem for $a=b=c=1$, $f(x,y)=1$, $g(x,y)=0$. Expected are displays for the **normal modes**, a **superposition** formula for $u(x,y,t)$, and **explicit numerical values** for the generalized Fourier coefficients.

4. (Finite String: Fourier Series Solution)
(a) [50%] Display the series formula, **complete with derivation details**, for the solution $u(x,t)$ of the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0<x<2, & t>0, \\ u(0,t) &= 0, & & t>0, \\ u(2,t) &= 0, & & t>0, \\ u(x,0) &= f(x), & 0<x<2, & \\ u_t(x,0) &= g(x), & 0<x<2. & \end{cases}$$

Symbols f and g should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display explicit formulas for the Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

(c) [25%] Evaluate the Fourier coefficients when $f(x)=100$ and $g(x)=0$.

Fourier Transform

1. (Fourier Transform Theory)
(a) [40%] Define Haberman's Fourier transform pair. Give an example of $f(x)$ and $F(w)$ which satisfy these equations.

(b) [60%] The heat equation on the line $-\infty<x<\infty$ can be solved by Fourier transform methods. Outline the method, called Fourier's Method, for the example

$$u_t = 4u_{xx}, \quad -\infty<x<\infty, \quad t>0, \quad u(x,0)=f(x).$$

2. (Fourier's Method)
Use the Heat kernel, the convolution theorem and the shift theorem to solve the diffusion-convection equation

$$u_t(x,t) = k u_{xx}(x,t) + c u_x(x,t), \quad t>0, \quad -\infty<x<\infty, \quad u(x,0)=f(x).$$

Answer: $u(x,t) = \dfrac{1}{\sqrt{4\pi k t}}\displaystyle\int_{-\infty}^{\infty} f(v) e^{\frac{-(x+ct-v)^2}{4kt}}\,dv$

3. (Heat Equation and Gauss' Heat Kernel)
Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \frac{1}{4}u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Hint: Use the heat kernel $g_t(x,v) = \dfrac{1}{\sqrt{kt}} e^{-\frac{(x-v)^2}{4kt}}$, the error function $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz$, and Fourier transform theory definitions to solve the problem. The answer is expressed in terms of the error function.