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Partial Differential Equations 3150

Midterm Exam 2

Exam Date: Wednesday, 9 April 2014

Instructions: This exam is timed for 50 minutes. Up to 60 minutes is possible. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

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100
100

Fourier Series

Problem 1. 100

A
-0

(a) [20%] State the **Fourier Convergence Theorem** for a function $f(x)$ defined on $-L \le x \le L$.

Graded Details: (1) Hypotheses, (2) Conclusion, (3) Series formula, (4) Coefficient formulas.

A
-0

(b) [20%] **True or False.** Each part earns 5/20 if answered correctly and 2/20 if answered incorrectly.

- T **True or False**. Assume $f(x)$ is periodic of period 5 and continuously differentiable on $-\infty < x < \infty$. Let $F(x)$ be the formal Fourier series of $f(x)$ on $|x| \le L$, $L = 5/2$. Then $f(11.2) = F(1.2)$.
- F **True or False**. Gibb's overshoot could fail to happen at a jump discontinuity, but when it happens the overshoot is about 9%.
- T **True or False**. The function $f(x) = \sin(2x) + \sin(\pi x)$ is odd but not periodic of any period.
- T **True or False**. The even periodic extension of $f(x) = x$ on $0 \le x \le 1$ of period 2 equals $|x - 6|$ on the interval $5 \le x \le 7$.

A

(c) [60%] Let $f(x)$ be the even extension to $|x| \le \pi$ of the function $\sin(2x)$ on $0 < x < \pi$. Then f has a formal **Fourier cosine series** $F(x)$ on $0 \le x \le \pi$, which is an even 2π-periodic extension of $f(x)$ to $-\infty < x < \infty$.

- Make a graph of $f(x)$ on $-\pi \le x \le \pi$.
- Make a graph of $F(x)$ over three periods.
- Write the series formula for $F(x)$ and the Fourier cosine coefficient formula.
- Find an integral formula, or the exact value, of the first **nonzero** term in the Fourier cosine series expansion $F(x)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Problem 1

a) $$\frac{f(x^-)+f(x^+)}{2} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

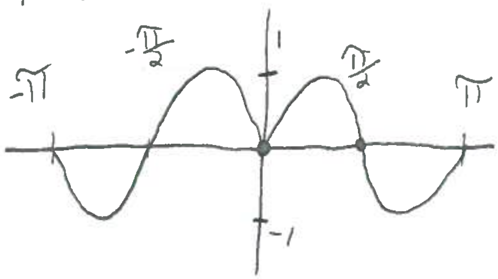
If $f(x)$ ~~has no discontinuities~~ is piecewise cont., then $\frac{f(x^-)+f(x^+)}{2} = f(x)$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

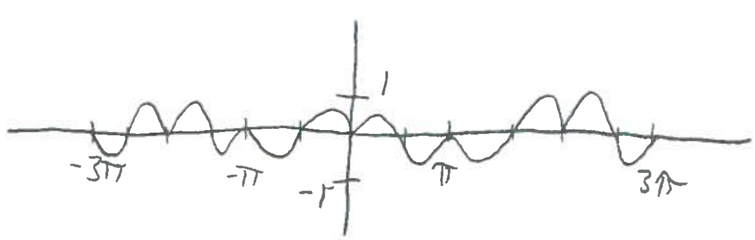
$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

b)
- True $\quad 1.2 + 5 + 5 = 11.2$
- False $\quad$ Gibbs' <u>always</u> at overshoot
- True
-  True

c) $f(x)$



$F(x)$



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$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right) \qquad L = \pi$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\,dx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin(2x)\cos(nx)\,dx$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} \sin(2x)\cos(x)\,dx \quad \leftarrow \text{First non zero term.}$$

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Wave Equation: The Finite String

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Problem 2.

A (a) [50%] Display the series formula, **complete with derivation details**, for the separation of variables solution $u(x,t)$ of the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \overset{c^2=1}{u_{xx}}(x,t), & 0<x<1, & t>0, \\ u(0,t) &= 0, & & t>0, \\ u(1,t) &= 0, & & t>0, \\ u(x,0) &= f(x), & 0<x<1, & \\ u_t(x,0) &= g(x), & 0<x<1. & \end{cases}$$

Expected in the formula for $u(x,t)$ are constants times product solutions (the normal modes).

Graded Details: (1) Separation of variables, (2) Product solution boundary value problem, (3) Product solution formulas,(4) Superposition details, (5) Series formula.

A (b) [20%] Display Fourier coefficient formulas for the solution of part (a).

A (c) [30%] Evaluate the Fourier coefficient formulas when $f(x)=0$ on $0\le x\le 1$ and $g(x)=50$ on $0\le x\le\frac{1}{2}$, $g(x)=0$ otherwise.

a) $u_{tt}(x,t) = (c^2)\, u_{xx}(x,t) \rightarrow u(x,t) = X(x)T(t)$

$XT'' = c^2 X'' T = -\lambda$

$\dfrac{T''}{c^2 T} = \dfrac{X''}{X} = -\lambda$

$\begin{cases} X''+\lambda X = 0 \\ X(0)=X(L)=0 \end{cases} \rightarrow \lambda>0 \quad X=\sin(\sqrt{\lambda}\,x)$

$\sqrt{\lambda}L = n\pi$

$\begin{cases} T''+c^2\lambda T = 0 \\ T\neq 0 \end{cases} \longrightarrow T=\cos\left(\frac{n\pi c t}{L}\right) = \sin\left(\frac{n\pi c t}{L}\right)$

normal modes: $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$, $\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$ where $L=1$

superposition:

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)\right)$$

where $L=1$

Use this page to start your solution. Attach extra pages as needed, then staple.

b) $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = a_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = a_n \int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)dx$$

$$a_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad \text{where } L = 1$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-a_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) + b_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)\right)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)$$

by same process:

$$b_m \frac{m\pi c}{L} = \frac{2}{L}\int_{-L}^{L} g(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad \text{where } L = 1$$

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

c) $f(x) = 0 \qquad 0 \le x \le 1$

$g(x) = 50 \qquad 0 \le x \le \frac{1}{2}$, $g(x) = 0$ otherwise

b/c $f(x) = 0$ and $a_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$a_n = 0$$

$$b_m = \frac{2}{m\pi c}\int_{-L}^{L} \begin{Bmatrix} 50 & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{Bmatrix} \sin\left(\frac{\pi x}{L}\right) dx$$

$$b_m = \frac{2}{m\pi}\int_0^{\frac{1}{2}} 50 \sin\left(\frac{m\pi x}{L}\right) dx$$

$(c^2 = 1)$

$$b_m = \frac{100}{m\pi}\left(\frac{L}{m\pi}\right)\left[-\cos\left(\frac{m\pi x}{L}\right)\right]_{x=0}^{x=\frac{1}{2}} \qquad L = 1$$

$$b_m = \frac{100}{(m\pi)^2}\left[-\cos\left(\frac{m\pi}{2}\right) + 1\right] \quad \text{ok}$$

$b_m = \frac{100}{(m\pi)^2}$ for $m =$ odd numbers

$b_m = \frac{100}{(m\pi)^2}\left[-\cos\left(\frac{m\pi}{2}\right) + 1\right]$ for $m =$ even numbers

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Fourier Transform

Problem 3.

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A (a) [10%] Define Haberman's Fourier transform pair.

A (b) [30%] Assume $f(x) = 2$ on $-1 \le x \le 0$ and $f(x) = 0$ otherwise. Compute the Fourier transform $F(w)$ of $f(x)$.

Graded Details: (1) Transform formula, (2) Integration details, (3) Answer.

A (c) [60%] The heat kernel $g(x)$ and the error function $\mathbf{erf}(x)$ are defined by the equations

$$g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}, \quad \mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Solve the infinite rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \frac{1}{16} u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Graded Details: (1) Fourier Transform method, (2) Convolution, (3) Heat kernel use, (4) Error function methods, (5) Final answer, expressed in terms of the error function.

a) $\begin{cases} F(\omega) = c_v \int_{-\infty}^{\infty} f(x) e^{i\omega x s} dx & \text{where } c_v = \frac{1}{2\pi} \text{ and } s = 1 \\ f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x s} d\omega \end{cases}$

b) $f(x) = \begin{cases} 2 & -1 \le x \le 0 \\ 0 & \text{else} \end{cases}$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{Bmatrix} 2 & -1 \le x \le 0 \\ 0 & \text{else} \end{Bmatrix} e^{i\omega x} dx$$

$$= \frac{1}{2\pi} \int_{-1}^{0} 2 e^{i\omega x} dx$$

$$= \frac{1}{\pi} \left[\frac{e^{i\omega x}}{i\omega} \right]_{-1}^{0}$$

$$F(\omega) = \frac{1}{\pi} \left(\frac{1 - e^{-i\omega x}}{i\omega} \right)$$

c) $g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$, $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\, dz$

$$\begin{cases} u_t(x,t) = \frac{1}{16} u_{xx}(x,t) & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \\ f(x) = \begin{cases} 50 & 0 < x < 1 \\ 0 & \text{else} \end{cases} \end{cases}$$

assume $\frac{1}{16} = k$

$FT[u_t] = k\, FT[u_{xx}]$

$\frac{d}{dt} FT[u] = k(i\omega)(i\omega)\, FT[u] = -k\omega^2 FT[u]$

if $U = FT[u]$ then have ODE:

$\frac{d}{dt} U = -k\omega^2 U$

$U = U_0 e^{-k\omega^2 t}$

solve for U_0:

$FT[u(x,t)] = U_0\, e^{-k\omega^2 t}$

$FT[u(x,0)] = U_0 e^0$

$FT[f(x)] = U_0 \longrightarrow U_0 = F(\omega)$

$U = F(\omega)\, e^{-k\omega^2 t}$

$\downarrow$ convolution theorem

$FT[u] = F(\omega)\, G(\omega)$ so $u = f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv$

$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{Bmatrix} 50 & 0 < x < 1 \\ 0 & \text{else} \end{Bmatrix} g(x-v)\, dx$

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$$u(x,t) = \frac{1}{2\pi}\int_0^1 50\, g(x-v)\, dx$$

$$u(x,t) = \frac{1}{2\pi}\int_0^1 50 \sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-v)^2}{4kt}}\, dx$$

change variables: $z^2 = \frac{(x-v)^2}{4kt} \rightarrow z = \frac{v-x}{\sqrt{4kt}}$ + $dz = \frac{dv}{\sqrt{4kt}}$

$$u(x,t) = \frac{1}{2\pi}\int_{v_1}^{v_2} 50 \sqrt{\frac{\pi}{\cancel{kt}}}\, e^{-z^2}\sqrt{4kt}\, dz$$

where $v_1 = \frac{0-x}{\sqrt{4kt}}$ + $v_2 = \frac{1-x}{\sqrt{4kt}}$

$$u(x,t) = 25\frac{2}{\sqrt{\pi}}\left(\int_0^{v_2} - \int_0^{v_1}\right)$$

$$u(x,t) = 25\left(\operatorname{erf}(v_2) - \operatorname{erf}(v_1)\right)$$

$v_2 = \frac{1-x}{\sqrt{4kt}}$, $v_1 = \frac{-x}{\sqrt{4kt}}$, $k = \frac{1}{16}$