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Math 3150 Midterm 1
Sample Exam

Problem 1. (Heat Conduction in a Rod, Ends at Different Temperatures)

Consider the heat conduction problem in a laterally insulated bar of length 1 with one end at zero Celsius and the other end at 100 Celsius. The initial temperature along the bar is given by function $f(x)$.

$$\begin{cases} u_t &= c^2 u_{xx}, & 0 < x < 1, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(1,t) &= 100, & & t > 0, \\ u(x,0) &= f(x), & 0 < x < 1. & \end{cases}$$

(a) [25%] Find the steady-state temperature $u_1(x)$.

(b) [50%] Solve the bar problem with zero Celsius temperatures at both ends, but $f(x)$ replaced by $f(x) - u_1(x)$. Call the answer $u_2(x,t)$. Besides the series answer for $u_2(x,t)$, which is a superposition of product solutions, please display the Fourier coefficient formula in integral form, unevaluated.

(c) [25%] Explain why the bar temperature is $u(x,t) = u_1(x) + u_2(x,t)$.

Problem 2. (Total Thermal Energy in a Rod)

If the temperature $u(x,t)$ is known, then give an expression for the time-dependent (because energy escapes at the ends) total thermal energy $\int_0^L e(x,t)\,A(x)\,dx$ contained in a rod $x = 0$ to $x = L$, with cross-sectional area $A(x)$. Symbol $e(x,t)$ is the thermal energy per unit volume at location x and time t, known to equal the specific heat times the mass density per unit volume times the temperature.

Validate the answer using a uniform rod of length L and constant cross-sectional area A, held at steady-state temperature $u = u_0$.

Problem 3. (Steady-State Heat Conduction on a Rectangular Plate)

Solve Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < L$, $0 < y < H$ subject to the boundary conditions. $u_x(x,y) = 0$ for $x = 0$ and $x = L$, $u(x,y) = 0$ for $y = 0$, $u(x,y) = f(x)$ for $y = H$.

Problem 4. (Poisson's Formula, Mean Value Theorem and the Maximum Principle)

For Laplace's equation inside a circular disk ($r < a$), the series solution formula can be re-arranged into Poisson's integral formula

$$u(r,\theta) = \int_0^{2\pi} f(\phi) K(r,\theta,\phi) d\phi, \quad \text{where}$$
$$K(r,\theta,\phi) = \frac{1}{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} = \text{Poisson's Kernel.}$$

Poisson's formula says that $u(r,\theta)$ is a weighted average of the boundary data $f(\theta)$ on the circle, with weight function K, the Poisson kernel.

(a) [50%] Compute K at $r = 0$, then show that $u(r,\theta)$ at $r = 0$ is the average value of f on the circle (Mean Value Theorem).

(b) [25%] Explain why solution $u = 100$ for $f(\theta) = 100$ does not contradict the Maximum Principle.

(c) [25%] Explain why K must have integral 1 over $0 \le \phi \le 2\pi$.

Problem 5. (Steady-State Heat Conduction on a Disk)

Consider the problem

$$\begin{cases} u_{rr}(r,\theta) + \dfrac{1}{r} u_r(r,\theta) + \dfrac{1}{r^2} u_{\theta\theta}(r,\theta) = 0, \quad 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(a,\theta) = f(\theta), \quad 0 < \theta < 2\pi. \end{cases}$$

Solve for $u(r,\theta)$ when $a = 1$ and $f(\theta) = 100$ on $0 \le \theta < \pi$, $f(\theta) = 0$ on $\pi \le \theta < 2\pi$.