Problem 1. (Heat Conduction in a Rod, Ends at Different Temperatures)
Throughout, $k$ is the mean thermal diffusivity, usually written as Fourier’s constant $K_0$ divided by specific heat $c$ and mass density per unit volume $\rho$.

(a) [40%] Consider the heat conduction problem in a laterally insulated rod of length 1 with one end at zero Celsius and the other end at 1 Celsius. The initial temperature along the rod is given by function $f(x) = x$.

$$
\begin{cases}
  u_t &= ku_{xx}, \quad 0 < x < 1, \quad t > 0, \\
  u(0,t) &= 0, \quad t > 0, \\
  u(1,t) &= 1, \quad t > 0, \\
  u(x,0) &= x, \quad 0 < x < 1.
\end{cases}
$$

The answer $u(x,t)$ to this problem is exactly the steady-state temperature. Find the answer $u(x,t)$ and display a complete answer check.

(b) [60%] Consider the heat conduction problem in a laterally insulated rod of length 1 with one end at zero Celsius and the other end at one Celsius. The initial temperature along the rod is given by function $f(x) = 1 + x$.

$$
\begin{cases}
  u_t &= ku_{xx}, \quad 0 < x < 1, \quad t > 0, \\
  u(0,t) &= 0, \quad t > 0, \\
  u(1,t) &= 1, \quad t > 0, \\
  u(x,0) &= 1 + x, \quad 0 < x < 1.
\end{cases}
$$

Solve the rod problem for $u(x,t)$. It is necessary to derive the product solutions. Provide Fourier coefficient formulas. Evaluate all Fourier coefficients. Then display the final answer $u(x,t)$.

(a) $u''_t = k u''_x = 0$

$$
\begin{align*}
  u &= c_1 + c_2 x \\
  u(0,t) &= 0 \\
  u(1,t) &= 1.
\end{align*}
$$

$c_1 = 0$, $c_2 = 1 \Rightarrow u(x,t) = x$

plug into $u_t = k u''_x$

$u_t = 0$

$u''_x = 0$

(b) $u(x,t) = \mathcal{X}(x) T(t)$

$$
\begin{align*}
  \mathcal{X} T' &= 0 \\
  \mathcal{X}'' + \lambda \mathcal{X} &= 0.
\end{align*}
$$

$T$, only $\lambda < 0$ makes sense

$$
T = e^{-\lambda t}.
$$

so $\mathcal{X} = e^{\lambda t}$, $\mathcal{X}(1) = 0$

only $\lambda > 0$ makes sense

$$
\mathcal{X} = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x), \quad \mathcal{X}(0) = 0
$$

$c_2 = 0$, $\sqrt{\lambda} = \kappa \pi$

next one...
problem 1(b) continued

Then  \( T = \sin(n\pi x) \) satisfies  \( T(0) = 0, T(1) = 0 \) and

\[
W(t) = e^{-n^2\pi^2 kt}
\]

from the product solutions for the ice-pack problem

\[
\begin{align*}
W(t) &= k \ W(x), \\
W(0, t) &= 0, \\
W(1, t) &= 0, \\
W(x, 0) &= (1 + x) - 2u_1(x) \\
&\quad = 1
\end{align*}
\]

\( u_1(x) = x \) is the steady-state from problem 1(a).

The solution to problem 2(b) is Then

\[
U(x, t) = W(x, t) + u_1(x)
\]

Because \( W(x, t) = \sum_{n=1}^{\infty} a_n \ \phi_n(x) \), then

\[
1 = W(x, 0) = \sum_{n=1}^{\infty} a_n \ \phi_n(x) e^0
\]

\[
a_n = \frac{1}{\phi_n(x) \ \phi_n(x)} = \int_0^1 \sin(n\pi x) \ dx
\]

\[
a_n = 2 \int_0^1 \sin(n\pi x) \ dx = 2 \left[ \frac{\cos(n\pi x)}{n\pi} \right]_{x=0}^{x=1}
\]

\[
a_n = 2 \left( \frac{(1)^n - 1}{n\pi} \right)
\]

Then

\[
U(x, t) = W(x, t) + u_1(x)
\]

\[
W(x, t) = \sum_{n=1}^{\infty} 2 \left( \frac{(1)^n - 1}{n\pi} \right) \sin(n\pi x) e^{-n^2\pi^2 kt} + x
\]
Problem 2. (Total Thermal Energy in a Rod)

An expression for the time-dependent total thermal energy contained in a rod $x = 0$ to $x = L$, with uniform cross-sectional area $A$ is

$$\int_0^L c p u(x, t) A \, dx \quad \text{J}$$

Symbol $c$ is the specific heat, $\rho$ is the mass density per unit volume and $u(x, t)$ is the rod temperature, satisfying the heat equation $u_t = k u_{xx}$. Assume $c$ and $\rho$ are constants.

Suppose $u(x, t)$ and $v(x, t)$ are two temperature distributions for the same rod which supply the same total thermal energy for all $t$.

A (a) [30%] Explain why $\int_0^L u(x, t) \, dx = \int_0^L v(x, t) \, dx$.

A (b) [60%] Differentiate on $t$ across the equation of (a). Simplify the resulting equation using $u_t = k u_{xx}$ and $v_t = k v_{xx}$ to obtain

$$u_x(L, t) - u_x(0, t) = v_x(L, t) - v_x(0, t).$$

A (c) [20%] Explain the meaning of the equation in (b) in terms of heat flux and Fourier's Law.

Total Thermal Energy = $\int_0^L c p A u(x, t) \, dx = \int_0^L c p A v(t) \, dx$.

a) This is because the rod is the same, so the values $c, \rho$, and $A$ cancel out. The two integrals must be the same because they supply the same total thermal energy for all $t$, even though they are different functions.

b) The temperature distributions $u(x, t)$ and $v(x, t)$ are related by

$$\frac{d}{dt} \left( \int_0^L u(x, t) \, dx \right) = \frac{d}{dt} \left( \int_0^L v(t) \, dx \right)$$

$$\int_0^L \frac{\partial}{\partial t} u(x, t) \, dx = \int_0^L \frac{\partial}{\partial t} v(x, t) \, dx$$

$$\int_0^L u_t(x, t) \, dx = \int_0^L v_t(x, t) \, dx$$

$$\int_0^L k u_{xx} \, dx = \int_0^L k v_{xx} \, dx$$

$$K \int_0^L u_x \, dx = K \int_0^L v_{xx} \, dx$$

$$u_x \bigg|_0^L = v_x \bigg|_0^L$$

$$u_x(L, t) - u_x(0, t) = v_x(L, t) - v_x(0, t)$$

This means that the difference between the heat flux at the right and left ends must equal the heat flux at the right and left ends for the $v(x, t)$ distribution.

c) Fourier's Law:

$$\phi = -K \frac{\partial u}{\partial t} \quad \text{heat flux}$$

This means that the heat flux at $x$ for $u(x, t)$ distribution must be equal to the heat flux at $x$ for $v(x, t)$ distribution.
Problem 2. (Total Thermal Energy in a Rod)

An expression for the time-dependent total thermal energy contained in a rod $x = 0$ to $x = L$, with uniform cross-sectional area $A$ is

$$\int_0^L c \rho u(x,t) A \, dx.$$

Symbol $c$ is the specific heat, $\rho$ is the mass density per unit volume and $u(x,t)$ is the rod temperature, satisfying the heat equation $u_t = ku_{xx}$. Assume $c$ and $\rho$ are constants.

Suppose $u(x,t)$ and $v(x,t)$ are two temperature distributions for the same rod which supply the same total thermal energy for all $t$.

A (a) [30%] Explain why $\int_0^L u(x,t) \, dx = \int_0^L v(x,t) \, dx$.

A (b) [60%] Differentiate on $t$ across the equation of (a). Simplify the resulting equation using $u_t = ku_{xx}$ and $v_t = kv_{xx}$ to obtain

$$u_t(L,t) - u_t(0,t) = v_t(L,t) - v_t(0,t).$$

A (c) [20%] Explain the meaning of the equation in (b) in terms of heat flux and Fourier’s Law.

\[ \text{If some total thermal energy supplied,} \]

$$\int_0^L c \rho u(x,t) A \, dx = \int_0^L c \rho v(x,t) A \, dx$$

Properties of rod are the same, cancel out.

$$\int_0^L u(x,t) \, dx = \int_0^L v(x,t) \, dx$$

b. \[ \frac{d}{dt} \int_0^L u(x,t) \, dx = \frac{d}{dt} \int_0^L v(x,t) \, dx \]

$$\int_0^L \frac{d}{dt} u(x,t) \, dx = \int_0^L \frac{d}{dt} v(x,t) \, dx$$

Since $u_t = ku_{xx}$, $v_t = kv_{xx}$.

$$K \int_0^L u_{xx}(x,t) \, dx = K \int_0^L v_{xx}(x,t) \, dx$$

Using the Fundamental Theorem of Calculus,

$$u_x(L,t) - u_x(0,t) = v_x(L,t) - v_x(0,t)$$
This indicates the heat flux, represented by $u_x$, and $v_x$, is the same across the length of the rod regardless of temperature distribution.

\[ \phi(x,t) = -K_0 u_x(x,t) \quad \text{(Fourier's Law)} \]

\[ K_0 (u_x(L,t) - u_x(0,t)) = -\phi(L,t) + \phi(0,t) \]

= total heat flux from
De ends $x = 0 \& \quad x = L$

Heat only escapes from De ends (lateral insulation).

The equation means the total heat flux from De ends is the same for $u$ and $v$, for each time $t > 0$. 

Problem 3. (Steady-State Heat Conduction on a Rectangular Plate)

Consider Laplace’s equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) on the rectangle \( 0 < x < 1, 0 < y < 1 \) subject to the boundary conditions
\( u_x(x,y) = 0 \) for \( x = 0 \) and \( x = 1 \), \( u(x,y) = 0 \) for \( y = 0 \) and \( y = 1 \), \( u(x,y) = f(x) \) for \( y = 0 \).

(a) [70%] Find the product solutions \( u(x,y) = X(x)Y(y) \). Include a check that each product solution satisfies the required three zero boundary conditions.

(b) [30%] Let \( f(x) \) be the sum of the first three eigenfunctions, which is the sum of the first three \( X \)-answers. Find \( u(x,y) \).

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{u_x(x,y)}{x} &= 0 \text{ for } x = 0, 1 \\
u_x(x,y) &= 0 \text{ for } x = 1 \\
u_x(x,y) &= f(x) \text{ for } y = 0 \\
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\frac{X''}{X} + \lambda X = 0 \\
X'(0) = X'(1) = 0 \\
Y'' - \lambda Y = 0 \\
Y(1) = 0
\end{cases}
\end{align*}
\]

\[
\lambda = 0 \Rightarrow Y = \begin{cases} 
0 \text{ for } \lambda = 0 \\
\end{cases}
\]

\[
X'' + \lambda X = 0
\]

\( X(0) = C_1 + C_2 \)
\( X'(0) = C_2 = 0 \Rightarrow C_2 = 0 \)
\( X'(1) = C_2 = 0 \Rightarrow C_2 = 0 \)
\( X(1) = C_1 \Rightarrow X = 1 \text{ for } \lambda = 0 \)

\( \lambda > 0 \Rightarrow Y(1) = 2e^{\sqrt{\lambda}y} + 2e^{-\sqrt{\lambda}y} = 0 \)
\( Y(1) = C_1e^{\sqrt{\lambda}y} + C_2e^{-\sqrt{\lambda}y} = 0 \)
\( Y(1) = C_1e^{\sqrt{\lambda}y} + C_2e^{-\sqrt{\lambda}y} = 0 \)

\[
Y(1) = \sinh(\sqrt{\lambda}(y-1))
\]

Correct!

Answer Checks on back.
Problem 3. (Steady-State Heat Conduction on a Rectangular Plate)

Consider Laplace's equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) on the rectangle \( 0 < x < 1, 0 < y < 1 \) subject to the boundary conditions
\( u_x(x, 0) = 0 \) for \( x = 0 \) and \( x = 1 \), \( u(x, y) = 0 \) for \( y = 0 \), \( u(x, y) = f(x) \) for \( y = 0 \).

(a) [70%] Find the product solutions \( u(x, y) = X(x)Y(y) \). Include a check that each product solution satisfies the required three boundary conditions.

(b) [30%] Let \( f(x) \) be the sum of the first three eigenfunctions, which is the sum of the first three \( X \)-answers. Find \( u(x, y) \).

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 < x < 1, \quad 0 < y < 1 \\
\frac{\partial u}{\partial x}(0, y) &= 0, \quad u(0, y) = 0, \quad u(1, y) = f(y) \\
\end{align*}
\]

\[
\begin{align*}
X'' + \lambda X &= 0, \\
X(0) &= X(1) = 0.
\end{align*}
\]

\[
\begin{align*}
\text{For } X &= 0, \quad \lambda = 0, \quad X = c_1 x \\
\text{For } \lambda > 0, \quad X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)
\end{align*}
\]

\[
\begin{align*}
Y'' - \lambda Y &= 0, \\
Y(0) &= 0, \quad Y(1) = 0.
\end{align*}
\]

\[
\begin{align*}
\text{For } \lambda = 0, \quad Y = c_1 y + c_2, \\
Y(0) = 0, \quad c_2 = 0, \quad Y = c_1 y, \quad Y(1) = c_1 = c_2, \quad Y = (1 - y) \\
\text{For } \lambda > 0, \quad Y = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}, \\
Y(0) = 0, \quad c_2 = 0, \quad Y = c_1 e^{\sqrt{\lambda} y} = \cosh(\sqrt{\lambda} y) \quad \text{good}
\end{align*}
\]

\[
\begin{align*}
\text{Ans: } Y &= y - 1 \quad (\lambda = 0) \\
Y &= \sinh(n\pi y - 1) \quad (\lambda > 0)
\end{align*}
\]

\[
\begin{align*}
\text{Ans: } X &= X_0 = a_0 (1 - y) \\
\text{For } \lambda = 0, \quad X(0) = 0, \quad X(1) = 0
\end{align*}
\]

\[
\begin{align*}
\text{Ans: } X_n &= a_n \cos(n\pi x) \cosh(n\pi y) \quad \text{Not zero}
\end{align*}
\]

\[
\begin{align*}
\text{Check, } \forall x = 0, 1; \cos(n\pi x) = 0, \quad \cosh(n\pi y) \quad \text{satisfied}
\end{align*}
\]

\[
\begin{align*}
\text{Check, } \forall y = 1; \cosh(n\pi y) \quad \text{satisfied}
\end{align*}
\]
Problem 3. (Steady-State Heat Conduction on a Rectangular Plate)

Consider Laplace's equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) on the rectangle \( 0 < x < 1, 0 < y < 1 \) subject to the boundary conditions 
\( u_x(x, y) = 0 \) for \( x = 0 \) and \( x = 1 \), \( u(x, y) = 0 \) for \( y = 0 \), \( u(x, y) = f(x) \) for \( y = 1 \).

(a) [70%] Find the product solutions \( u(x, y) = X(x)Y(y) \). Include a check that each product solution satisfies the required three zero boundary conditions.

(b) [30%] Let \( f(x) \) be the sum of the first three eigenfunctions, which is the sum of the first three \( X \)-answers. Find \( u(x, y) \).

\[
\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0
\]

\[
X'' + \gamma X = 0
\]

BC: \( X(0) = 0 \)
\( X(1) = 0 \)

\[
Y'' + \lambda Y = 0
\]

BC: \( Y(0) = 0 \)
\( Y(1) = 0 \)

\[
\lambda > 0
\]

\[
X = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x
\]

\[
Y = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}
\]

\[
\gamma = \sinh \lambda \lambda
\]

\[
\text{Correction: } \gamma = \sinh (\sqrt{\lambda} (y - 1))
\]

**Superposition**

\[
\lambda > 0 \quad \lambda = 0
\]

\[
X = \cos \sqrt{\lambda} x \quad x = 1
\]

\[
Y = \sinh (\sqrt{\lambda} (y - 1)) \quad y = 1
\]

**Apply Correction**

Orthogonality (broken)

\[
\int_0^1 X(x)Y(x) dx = 0
\]

\[
\int_0^1 (x) (1) = (0)^0 + \sum_{n=1}^\infty \cos \frac{n\pi y}{a} y
\]

**Correction**

\[
\int_0^a f(x) \cos nx dx
\]

\[
am = \frac{2}{a} \int_0^a f(x) \cos nx dx
\]
Problem 36

\[ u(x, y) = a_0 (1-y) + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \cosh(n \pi y) \]

\[ f(x) = 1 + \cos(\pi x) + \cos(2\pi x) \quad \text{(good)} \]

\[ u(x, 0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \]

\[ a_0 = \frac{\int_0^1 f(x) \, dx}{\int_0^1 \cos(0 \pi x) \, dx} \]

\[ \sinh(-n \pi) a_n = \frac{\int_0^1 f(x) \cos(n \pi x) \, dx}{\int_0^1 \cos^2(n \pi x) \, dx} \quad n \geq 1 \]

\[ u(x, y) = a_0 (1-y) + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \cosh(n \pi y) \]

Please, insert y-terms into \( f(x) \). NO Fourier coefficient formula is needed.

**Solution**

\[ u(x, y) = 1 \cdot \frac{Y_0(y)}{Y_0(0)} + \cos(\pi x) \cdot \frac{Y_1(y)}{Y_1(0)} + \cos(2\pi x) \cdot \frac{Y_2(y)}{Y_2(0)} \]

\[ Y_0 = y-1, \quad Y_1 = \sinh(\pi(y-1)), \quad Y_2 = \sinh(2\pi(y-1)) \]

Then

\[ u(x, 0) = 1 + \cos(\pi x) + \cos(2\pi x) = f(x) \]
Problem 4. (Steady-State Heat Conduction on a Disk)

Consider the steady-state heat conduction problem in polar coordinates

\[
\begin{align*}
u_{rr}(r, \theta) + \frac{1}{r} u_r(r, \theta) + \frac{1}{r^2} u_{\theta\theta}(r, \theta) &= 0, \quad 0 < r < 2, \quad 0 < \theta < 2\pi, \\
u(2, \theta) &= f(\theta), \quad 0 < \theta < 2\pi.
\end{align*}
\]

(a) [60%] Find the product solutions \( u = R(r)\Theta(\theta) \), then identify the orthogonal set and the interval. Stop at this stage: omit superposition, omit the series solution and do not develop formulas for the Fourier coefficients.

(b) [40%] Calculate \( u(0, \theta) \) when \( f(\theta) = 0 \) on \( 0 \leq \theta < \pi \), \( f(\theta) = 50 \) on \( \pi \leq \theta < 2\pi \). Hint: The Poisson integral theorem and the Mean Value Theorem.

(a) Product solutions:

\[
\begin{align*}
U_0(r, \theta) &= a_0 \\
U_n(r, \theta) &= \left[ a_n \cos(n\pi) + b_n \sin(n\pi) \right] r^n
\end{align*}
\]

Orthogonal set: \( \{ 1, \cos(n\pi), \sin(n\pi) \} \)

Interval of: \( 0 < \theta < 2\pi \)

(b) Mean value theorem states: \( u(r, \theta) = \frac{1}{\theta \pi} \int_0^{2\pi} f(\theta) d\theta \), where \( f(\theta) \) is the temperature distribution along the edge of the disk.

\[
u(0, \theta) = \frac{1}{\theta \pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{\theta \pi} \int_0^{2\pi} (50 \pi) d\theta = \frac{1}{\theta \pi} (50 \pi \theta) = 25
\]

\[\boxed{\nu(0, \theta) = 25}\]